

POSITIVE DIVISORS IN SYMPLECTIC GEOMETRY

JIANXUN HU & YONGBIN RUAN

CONTENTS

1. Introduction	1
2. Preliminaries	4
2.1. GW-invariants	4
2.2. Relative GW-invariants	9
2.3. Partial orderings on relative GW invariants	13
2.4. Degeneration formula	14
3. Relative GW-invariants of \mathbb{P}^1 -bundles	17
3.1. Fiber class invariants	17
3.2. A vanishing theorem	21
3.3. A nonvanishing theorem	25
4. A comparison theorem	26
5. Rationally connected symplectic divisors	28
5.1. Rationally connectedness in algebraic geometry	28
5.2. Rationally connected symplectic divisors	29
References	33

1. INTRODUCTION

Divisors or codimension two symplectic submanifolds play an important role in Gromov-Witten theory and symplectic geometry. For example, there is now a well-known degeneration operation to decompose a symplectic manifold X into $X_1 \cup_Z X_2$ a union of X_1, X_2 , along a common divisor Z . To utilize the above degeneration to compute Gromov-Witten invariants inductively, one needs to develop the relative Gromov-Witten theory of the pair (X, Z) . Such a theory and its degeneration formula were first constructed by Li-Ruan [LR] (see Ionel and Parker [IP] for a different version and J. Li [Li1, Li2] for an algebraic treatment).

Maulik and Pandharipande [MP] systematically studied the degeneration formula for the degeneration $X \rightarrow X \cup_Z P_Z$, where P_Z is the projective closure of the normal bundle N_Z . By introducing a certain partial order on

¹Partially supported by the NSFC Grant 10631050, NKBRPC(2006CB805905) and NCET-04-0795.

³supported by NSF Grant.

relative invariants, they interpreted the degeneration formula as a “correspondence”, a complicated upper triangular linear map from relative invariants to absolute invariants. One consequence of their correspondence is that the absolute and relative invariants determine each other. Such a correspondence is a very powerful tool in determining the totality of Gromov-Witten theory. But it is not effective in determining any single invariant. A natural question is if a divisor with a stronger condition will give us a much stronger correspondence? We answer this affirmatively for so-called positive symplectic divisors. We call a symplectic divisor Z *positive* if for some tamed almost complex structure J , $C_1(N_Z)(A) > 0$ for any A represented by a non-trivial J -sphere in Z . This is a generalization of ample divisor from algebraic geometry. One of our main theorems is the following comparison theorem.

We call $0 \neq A \in H_2(Z, \mathbb{Z})$ *stably effective* if there is a nonzero genus zero Gromov-Witten invariant of Z with class A . Let $\pi : P_Z \rightarrow Z$ be the projection. Maulik and Pandharipande show that any nonzero absolute or relative Gromov-Witten invariant of P_Z is determined by the Gromov-Witten invariants of Z . It is a direct consequence of their argument that if a nonzero genus zero Gromov-Witten or relative Gromov-Witten invariant of P_Z has class B , then $\pi_*(B) = \sum_i a_i A_i$ for stably effective $A_i \in H_2(Z, \mathbb{Z})$ and $a_i \in \mathbb{Z}_+$. Let $V = \min \{C_1(N_Z(A)) \mid A \in H_2(Z, \mathbb{Z}) \text{ is stably effective}\}$.

Theorem 1.1. *Suppose that Z is a positive divisor and $V \geq l$. Then for $A \in H_2(X, \mathbb{Z})$, $\alpha_i \in H^*(X, \mathbb{R})$, $1 \leq i \leq \mu$, and $\beta_j \in H^*(Z, \mathbb{R})$, $1 \leq j \leq l$, we have*

$$\begin{aligned} & \langle \alpha_1, \dots, \alpha_\mu, \iota^!(\beta_1), \dots, \iota^!(\beta_l) \rangle_A^X \\ &= \sum_{\mathcal{T}} \langle \alpha_1, \dots, \alpha_\mu \mid \mathcal{T} \rangle_A^{X, Z}, \end{aligned}$$

where $\iota : Z \hookrightarrow X$ and $\iota^! = PD \iota_* PD$, the summation runs over all possible weighted partitions $\mathcal{T} = \{(1, \gamma_1), \dots, (1, \gamma_q), (1, [Z]), \dots, (1, [Z])\}$ of $Z \cdot A$, where γ_i 's are the products of some β_j classes.

McDuff [M1] also considered the similar comparison result in the some special case. For readers familiar with Maulik-Pandharipande's relative/absolute correspondence, the above theorem means that there is no lower order term in the degeneration formula.

The second motivation comes from symplectic birational geometry. A fundamental problem in symplectic geometry is to generalize birational geometry to symplectic geometry. In the 80's, Mori introduced a program towards the birational classification of algebraic manifolds of dimension three and up. In 90's, the last author [R1] speculated that there should be a symplectic geometric program. First of all, since there is no notion of regular or rational function in symplectic geometry, we must therefore first define the appropriate notion of symplectic birational equivalence. For this purpose, in [HLR], the authors proposed to use Guillemin-Sternberg's birational

cobordism to replace birational maps. Secondly, we need to study what geometric properties behave nicely under this birational cobordism. In [HLR], the authors defined the uniruledness of symplectic manifolds by requiring a nonzero Gromov-Witten invariant with a point insertion and settled successfully the fundamental birational cobordism invariance of uniruledness. McDuff [M2] proved that Hamiltonian S^1 -manifolds are uniruled using the techniques from [HLR].

Furthermore, Tian-Jun Li and the second author [LtjR] investigate the dichotomy of uniruled symplectic divisors. The dichotomy asserts that if the normal bundle is non-negative in some sense, then the ambient manifold is uniruled.

It is clear that our stronger comparison theorem should yield stronger results. This is indeed the case. As an application of our comparison theorem, we investigate symplectic rationally connected manifolds. Similar to the case of uniruledness (see [HLR]), we define the notion of k -point (strongly) rational connectedness by requiring a non-zero (primary) Gromov-Witten invariant with k point insertions (see section five for the detailed discussion). Of course, one important problem here of interest is whether the notion of symplectic k -point rational connectedness is invariant under birational cobordism given in [HLR]. From the blowup formula of [H1, H2, H3, HZ, La], we know that symplectic rational connectedness is invariant under the symplectic blowup along points and some special submanifolds with convex normal bundles. The general case is still unknown. We should mention that a longstanding problem in Gromov-Witten theory is to characterize algebraic rationally connectedness in terms of Gromov-Witten theory. Our second main theorem is the following theorem analogous to the theorem of McDuff [M3, LtjR] for uniruled divisors.

Theorem 1.2. *Let (X, ω) be a compact $2n$ dimensional symplectic manifold which contains a submanifold P symplectomorphic to \mathbb{P}^{n-1} whose normal Chern number $m \geq 2$, then X is strongly rationally connected.*

The paper is organized as follows. In section two, we first review Gromov-Witten theory and its degeneration formula to set up the notation. In Section three, we prove some vanishing and non-vanishing results for relative Gromov-Witten invariants of \mathbb{P}^1 -bundles. In section four, we prove a comparison theorem between absolute and relative Gromov-Witten invariants. In section five, we generalize the *from divisor to ambient space* inductive construction of [LtjR] to the case k -point rational connectedness.

Acknowledgements: Both of the authors would like to thank Zhenbo Qin who took part in the early time of this work. This article was prepared during the first author's visit to the Department of Mathematics, University of Michigan-Ann Arbor and the Department of Mathematics, University of Missouri-Columbia. The first author is grateful to the departments for

their hospitality. The authors would like to thank Tian-Jun Li for his valuable suggestions and telling us about McDuff's work. Thanks also to A. Greenspoon for his editorial assistance.

2. PRELIMINARIES

In this section, we want to review briefly the constructions of virtual integration in the definitions of the absolute and relative GW invariants, which are the main tool of our paper. We refer to [R1, LR] for the details.

2.1. GW-invariants. Suppose that (X, ω) is a compact symplectic manifold and J is a tamed almost complex structure.

Definition 2.1. *A stable J -holomorphic map is an equivalence class of pairs (Σ, f) . Here Σ is a connected nodal marked Riemann surface with arithmetic genus g , k smooth marked points x_1, \dots, x_k , and $f : \Sigma \rightarrow X$ is a continuous map whose restriction to each component of Σ (called a component of f in short) is J -holomorphic. Furthermore, it satisfies the stability condition: if $f|_{S^2}$ is constant (called a ghost bubble) for some S^2 -component, then the S^2 -component has at least three special points (marked points or nodal points). (Σ, f) , (Σ', f') are equivalent, or $(\Sigma, f) \sim (\Sigma', f')$, if there is a biholomorphic map $h : \Sigma' \rightarrow \Sigma$ such that $f' = f \circ h$.*

An essential feature of Definition 2.1 is that, for a stable J -holomorphic map (Σ, f) , the automorphism group

$$\text{Aut}(\Sigma, f) = \{h \mid h \circ (\Sigma, f) = (\Sigma, f)\}$$

is finite. We define the moduli space $\overline{\mathcal{M}}_A^X(g, k, J)$ to be the set of equivalence classes of stable J -holomorphic maps such that $[f] = f_*[\Sigma] = A \in H_2(X, \mathbf{Z})$. The virtual dimension of $\overline{\mathcal{M}}_A^X(g, k, J)$ is computed by index theory,

$$\text{vir} \dim_{\mathbb{R}} \overline{\mathcal{M}}_A^X(g, k, J) = 2c_1(A) + 2(n-3)(1-g) + 2k,$$

where n is the complex dimension of X .

Unfortunately, $\overline{\mathcal{M}}_A^X(g, k, J)$ is highly singular and may have larger dimension than the virtual dimension. To extract invariants, we use the following virtual neighborhood method.

First, we drop the J -holomorphic condition from the previous definition and require only that each component of f be smooth. We call the resulting object a stable map or a C^∞ -stable map. Denote the corresponding space of equivalence classes by $\overline{\mathcal{B}}_A^X(g, k, J)$. $\overline{\mathcal{B}}_A^X(g, k, J)$ is clearly an infinite dimensional space. It has a natural stratification given by the topological type of Σ together with the fundamental classes of the components of f . The stability condition ensures that $\overline{\mathcal{B}}_A^X(g, k, J)$ has only finitely many strata such that each stratum is a Fréchet orbifold. Further, one can use the pregluing construction to define a topology on $\overline{\mathcal{B}}_A^X(g, k, J)$ which is Hausdorff and makes $\overline{\mathcal{M}}_A^X(g, k, J)$ a compact subspace. (see [R1]).

We can define another infinite dimensional space $\Omega^{0,1}$ together with a map

$$\pi : \Omega^{0,1} \longrightarrow \overline{\mathcal{B}}_A^X(g, k, J)$$

such that the fiber is $\pi^{-1}(\Sigma, f) = \Omega^{0,1}(f^*TX)$. The Cauchy-Riemann operator is now interpreted as a section of $\pi : \Omega^{0,1} \longrightarrow \overline{\mathcal{B}}_A^X(g, k, J)$,

$$\overline{\partial}_J : \overline{\mathcal{B}}_A^X(g, k, J) \rightarrow \Omega^{0,1}$$

with $\overline{\partial}_J^{-1}(0)$ nothing but $\overline{\mathcal{M}}_A^X(g, k, J)$.

At each $(\Sigma, f) \in \overline{\mathcal{M}}_A^X(g, k, J)$, there is a canonical decomposition of the tangent space of $\Omega^{0,1}$ into the horizontal piece and the vertical piece. Thus we can linearize $\overline{\partial}_J$ with respect to deformations of stable maps and project to the vertical piece to obtain an elliptic complex

$$(1) \quad L_{\Sigma, f} : \Omega^0(f^*TX) \longrightarrow \Omega^{0,1}(f^*TX).$$

Several explanations are in order for formula (1). Choose a compatible Riemannian metric on X and let ∇ be the Levi-Civita connection.

When Σ is irreducible, ∇ induces a connection on f^*TX , still denoted by ∇ . Then $L_{\Sigma, f} = \overline{\nabla}$, where $\overline{\nabla}$ is the projection of ∇ onto the $(0, 1)$ -factor.

When Σ is reducible, formula (1) is interpreted as follows. For simplicity, suppose that Σ is the union of Σ_1 and Σ_2 intersecting at $p \in \Sigma_1$ and $q \in \Sigma_2$. Let the corresponding maps be f_1 and f_2 . Then define

$$\Omega^0(f^*TX) = \{(v_1, v_2) \in \Omega^0(f_1^*TX) \times \Omega^0(f_2^*TX) \mid v_1(p) = v_2(q)\}$$

and

$$\Omega^{0,1}(f^*TX) = \Omega^{0,1}(f_1^*TX) \oplus \Omega^{0,1}(f_2^*TX).$$

$L_{\Sigma, f}$ is then the restriction of $L_{\Sigma_1, f_1} \oplus L_{\Sigma_2, f_2}$ to Ω^0 . It is clear that

$$\text{Ker } L_{\Sigma, f} = \{(v_1, v_2) \in \text{Ker } L_{\Sigma_1, f_1} \times \text{Ker } L_{\Sigma_2, f_2} \mid v_1(p) = v_2(q)\}.$$

To understand $\text{Coker } L_{\Sigma, f}$ geometrically, it is convenient to use another elliptic complex. The idea is as follows. We would like to drop the condition $v_1(p) = v_2(q)$. To keep the index unchanged, we need to enlarge $\Omega^{0,1}(f^*TX)$. A standard method motivated by algebraic geometry is to allow a simple pole at the intersection point. This leads to

$$\tilde{L}_{\Sigma, f} : \tilde{\Omega}^0(f^*TX) \longrightarrow \tilde{\Omega}^{0,1}(f^*TX),$$

where

$$\begin{aligned} \tilde{\Omega}^0(f^*TX) &= \Omega^0(f_1^*TX) \times \Omega^0(f_2^*TX), \\ \tilde{\Omega}^{0,1}(f^*TX) &= \{(v_1, v_2) \in \Omega^{0,1}(f_1^*TX \otimes \mathcal{O}(p)) \times \Omega^{0,1}(f_2^*TX \otimes \mathcal{O}(q)) \mid \\ &\quad \text{Res}_p v_1 + \text{Res}_q v_2 = 0\}, \end{aligned}$$

and

$$\tilde{L}_{\Sigma, f} = \tilde{L}_{\Sigma_1, f_1} \oplus \tilde{L}_{\Sigma_2, f_2}.$$

It is well-known that

$$\text{Ker } L_{\Sigma, f} \cong \text{Ker } \tilde{L}_{\Sigma, f}, \quad \text{Coker } L_{\Sigma, f} \cong \text{Coker } \tilde{L}_{\Sigma, f}.$$

Therefore

$$\begin{aligned} \text{Coker } L_{\Sigma, f} = & \{(v_1, v_2) \in \text{Coker } \tilde{L}_{\Sigma_1, f_1} \times \text{Coker } \tilde{L}_{\Sigma_2, f_2} \mid \\ & \text{Res}_p v_1 + \text{Res}_q v_2 = 0\}. \end{aligned}$$

If Σ has more than two components, the construction above extends in a straightforward fashion.

To consider the full linearization of $\bar{\partial}_J$, we have to include the deformation space $Def(\Sigma)$ of the nodal marked Riemann surface Σ . $Def(\Sigma)$ fits into the short exact sequence,

$$0 \longrightarrow H^1(T\Sigma) \longrightarrow Def(\Sigma) \longrightarrow T_p\Sigma_1 \otimes T_q\Sigma_2 \longrightarrow 0,$$

where the first term represents the deformation space of Σ preserving the nodal point and the third term represents the smoothing of the nodal point. Moreover, $H^1(T\Sigma)$ is a product, with each factor being the deformation space of a component while treating the nodal point as a new marked point.

The full linearization of $\bar{\partial}_J$ is given by $L_{\Sigma, f} \oplus \frac{1}{2}Jdf$. Denote

$$\begin{aligned} Def(\Sigma, f) &= \text{Ker}(L_{\Sigma, f} \oplus \frac{1}{2}Jdf)/T_e\text{Aut}(\Sigma), \\ Obs(\Sigma, f) &= \text{Coker}(L_{\Sigma, f} \oplus \frac{1}{2}Jdf). \end{aligned}$$

When $Obs(\Sigma, f) = 0$, (Σ, f) is a smooth point of the moduli space and $Def(\Sigma, f)$ is its tangent space.

Now we choose a nearby symplectic form ω' such that ω' is tamed with J and $[\omega']$ is a rational cohomology class. Using ω' , B. Siebert [S1] (see also the appendix in [R1]) constructed a natural finite dimensional vector bundle over $\bar{\mathcal{B}}_A(g, k, J)$. It has the property of dominating any local finite dimensional orbifold bundle as follows. Let U be a neighborhood of $(\Sigma, f) \in \bar{\mathcal{B}}_A(g, k, J)$ and F_U be an orbifold bundle over U . Then Siebert constructed a bundle E over $\bar{\mathcal{B}}_A(g, k, J)$ such that there is a surjective bundle map $E|_U \longrightarrow F_U$.

For each $(\Sigma, f) \in \bar{\mathcal{M}}_A(g, k, J)$, $Obs(\Sigma, f)$ extends to a local orbifold bundle $F(\Sigma, f)$ over a neighborhood $\tilde{\mathcal{U}}_{\Sigma, f}$ of (Σ, f) . Then we use Siebert's construction to find a global orbifold bundle $\mathcal{E}(\Sigma, f)$ dominating $F(\Sigma, f)$. In fact, any global orbifold bundle with this property will work. We also remark that it is often convenient to replace $Obs(\Sigma, f)$ by $\text{Coker } L_{\Sigma, f}$ in the construction. Over each $\tilde{\mathcal{U}}_{\Sigma, f}$, by the domination property, we can construct a stabilizing term $\eta_{\Sigma, f} : \mathcal{E}(\Sigma, f) \longrightarrow \Omega^{0,1}$ supported in $\tilde{\mathcal{U}}_{\Sigma, f}$ such that $\eta_{\Sigma, f}$ is surjective onto $Obs(\Sigma, f)$ at (Σ, f) . Obviously, $\eta_{\Sigma, f}$ can be viewed as a map from $\mathcal{E}(\Sigma, f)$ to $\Omega^{0,1}$. Then the stabilizing equation

$$\bar{\partial}_J + \eta_{\Sigma, f} : \mathcal{E}(\Sigma, f) \rightarrow \Omega^{0,1}, \quad ((\Sigma', f'), e) \rightarrow \bar{\partial}_J f' + \eta_{\Sigma, f}(e)$$

has no cokernel at (Σ, f) . By semicontinuity, it has no cokernel in a neighborhood $\mathcal{U}_{\Sigma, f} \subset \tilde{\mathcal{U}}_{\Sigma, f}$.

Since $\overline{\mathcal{M}}_A^X(g, k, J)$ is compact, there are finitely many $\mathcal{U}_\gamma = \mathcal{U}_{\Sigma_\gamma, f_\gamma}$ covering $\overline{\mathcal{M}}_A^X(g, k, J)$. Let

$$\mathcal{U} = \cup_\gamma \mathcal{U}_\gamma, \quad \mathcal{E} = \oplus_\gamma \mathcal{E}(\Sigma_\gamma, f_\gamma), \quad \eta = \sum_\gamma \eta_{\Sigma_\gamma, f_\gamma}.$$

Consider the finite dimensional vector bundle over \mathcal{U} , $p : \mathcal{E}|_{\mathcal{U}} \rightarrow \mathcal{U}$. The stabilizing equation $\overline{\partial}_J + \eta$ can be interpreted as a section of the bundle $p^*\Omega^{0,1} \rightarrow \mathcal{E}|_{\mathcal{U}}$. By construction, this section

$$\overline{\partial}_J + \eta : \mathcal{E}|_{\mathcal{U}} \rightarrow p^*\Omega^{0,1}$$

is transverse to the zero section of $p^*\Omega^{0,1} \rightarrow \mathcal{E}|_{\mathcal{U}}$.

The set $U_{\mathcal{S}_e}^X = (\overline{\partial}_J + \eta)^{-1}(0)$ is called the virtual neighborhood in [R1]. The heart of [R1] is to show that $U_{\mathcal{S}_e}^X$ has the structure of a C^1 -manifold.

Notice that $U_{\mathcal{S}_e}^X \subset \mathcal{E}|_{\mathcal{U}}$. Over $U_{\mathcal{S}_e}^X$ there is the tautological bundle

$$\mathcal{E}_X = p^*(\mathcal{E}|_{\mathcal{U}})|_{U_{\mathcal{S}_e}^X}.$$

It comes with the tautological inclusion map

$$S_X : U_{\mathcal{S}_e}^X \rightarrow \mathcal{E}_X, \quad ((\Sigma', f'), e) \rightarrow e,$$

which can be viewed as a section of \mathcal{E}_X . It is easy to check that

$$S_X^{-1}(0) = \overline{\mathcal{M}}_A^X(g, k, J).$$

Furthermore, one can show that S_X is a proper section.

Note that the stratification of $\overline{\mathcal{B}}_A(g, k, J)$ induces a natural stratification of \mathcal{E} . We can define $\eta_\gamma = \eta_{\Sigma_\gamma, f_\gamma}$ inductively from lower stratum to higher stratum. For example, we can first define η_γ on a stratum and extend to a neighborhood. Then we define $\eta_{\gamma+1}$ at the next stratum supported away from lower strata. One consequence of this construction is that $U_{\mathcal{S}_e}^X$ has the same stratification as that of \mathcal{E} . Namely, if $\mathcal{B}_{D'} \subset \overline{\mathcal{B}}_D$ is a lower stratum,

$$U_{\mathcal{S}_e}^X \cap \mathcal{E}|_{\mathcal{B}_{D'}} \subset U_{\mathcal{S}_e}^X \cap \mathcal{E}|_{\overline{\mathcal{B}}_D}$$

is a submanifold of codimension at least 2.

There are evaluation maps

$$ev_i : \overline{\mathcal{B}}_A(g, k, J) \rightarrow X, \quad (\Sigma, f) \rightarrow f(x_i),$$

for $1 \leq i \leq k$. ev_i induces a natural map from $U_{\mathcal{S}_e} \rightarrow X^k$, which can be shown to be smooth.

Let Θ be the Thom form of the finite dimensional bundle $\mathcal{E}_X \rightarrow U_{\mathcal{S}_e}^X$.

Definition 2.2. *The (primary) Gromov-Witten invariant is defined as*

$$\langle \alpha_1, \dots, \alpha_k \rangle_{g,A}^X = \int_{U_{\mathcal{S}_e}} S_X^* \Theta \wedge \Pi_i ev_i^* \alpha_i,$$

where $\alpha_i \in H^*(X; \mathbb{R})$. For the genus zero case, we also write $\langle \alpha_1, \dots, \alpha_k \rangle_A^X =$

$$\langle \alpha_1, \dots, \alpha_k \rangle_{0,A}^X.$$

Definition 2.3. For each marked point x_i , we define an orbifold complex line bundle \mathcal{L}_i over $\overline{\mathcal{B}}_A^X(g, k, J)$ whose fiber is $T_{x_i}^*\Sigma$ at (Σ, f) . Such a line bundle can be pulled back to $U_{\mathcal{S}_e}^X$ (still denoted by \mathcal{L}_i). Denote $c_1(\mathcal{L}_i)$, the first Chern class of \mathcal{L}_i , by ψ_i .

Definition 2.4. The descendent Gromov-Witten invariant is defined as

$$\langle \tau_{d_1}\alpha_1, \dots, \tau_{d_k}\alpha_k \rangle_{g,A}^X = \int_{U_{\mathcal{S}_e}^X} S_X^* \Theta \wedge \Pi_i \psi_i^{d_i} \wedge ev_i^* \alpha_i,$$

where $\alpha_i \in H^*(X; \mathbb{R})$.

Remark 2.5. In the stable range $2g + k \geq 3$, one can also define non-primary GW invariants (See e.g. [R1]). Recall that there is a map $\pi : \overline{\mathcal{B}}_A^X(g, k, J) \rightarrow \overline{\mathcal{M}}_{g,k}$ contracting the unstable components of the source Riemann surface. We can introduce a class κ from the Deligne-Mumford space via π to define the ancestor GW invariants

$$\langle \kappa \mid \Pi_i \alpha_i \rangle_{g,A}^X = \int_{U_{\mathcal{S}_e}^X} S_X^* \Theta \wedge \pi^* \kappa \wedge \Pi_i ev_i^* \alpha_i.$$

The primary Gromov-Witten invariants are the special invariants with the point class in $\overline{\mathcal{M}}_{0,k}$.

Remark 2.6. For computational purpose we would mention the following variation of the virtual neighborhood construction. Suppose $\iota : D \subset X$ is a submanifold. For $\alpha \in H^*(D; \mathbb{R})$ we define $\iota^!(\alpha) \in H^*(X; \mathbb{R})$ via the transfer map $\iota^! = PD_X \circ \iota_* \circ PD_D$. One can construct Gromov-Witten invariants with an insertion of the form $\iota^!(\alpha)$ as follows. Apply the virtual neighborhood construction to the compact subspace

$$\overline{\mathcal{M}}_A(g, k, J) \cap ev_1^{-1}(D)$$

in $\overline{\mathcal{B}}_A^X(g, k, J, D) = ev_1^{-1}(D)$ to obtain a virtual neighborhood $\mathcal{U}_{\mathcal{S}_e}(D)$ together with the natural map $ev_D : \mathcal{U}_{\mathcal{S}_e}(D) \rightarrow D$. It is easy to show that

$$\langle \tau_{d_1} \iota^!(\alpha), \tau_{d_2} \beta_2, \dots, \tau_{d_k} \beta_k \rangle_{g,A}^X = \int_{\mathcal{U}_{\mathcal{S}_e}(D)} S^* \Theta \wedge ev_D^* \alpha \wedge \prod_{i=2}^k \psi_i^{d_i} ev_i^* \beta_i.$$

Remark 2.7. For each $\langle \tau_{d_1} \alpha_1, \dots, \tau_{d_k} \alpha_k \rangle_{g,A}^X$, we can conveniently associate a simple graph Γ of one vertex decorated by (g, A) and a tail for each marked point. We then further decorate each tail by (d_i, α_i) and call the resulting graph $\Gamma(\{(d_i, \alpha_i)\})$ a weighted graph. Using the weighted graph notation, we denote the above invariant by $\langle \Gamma(\{(d_i, \alpha_i)\}) \rangle^X$. We can also consider the disjoint union Γ^\bullet of several such graphs and use $A_{\Gamma^\bullet}, g_{\Gamma^\bullet}$ to denote the total homology class and total arithmetic genus. Here the total arithmetic genus is $1 + \sum (g_i - 1)$. Then, we define $\langle \Gamma^\bullet(\{(d_i, \alpha_i)\}) \rangle^X$ as the product of Gromov-Witten invariants of the connected components.

2.2. Relative GW-invariants. In this section, we will review the relative GW-invariants. The readers can find more details in the reference [LR].

Let $Z \subset X$ be a real codimension 2 symplectic submanifold. Suppose that J is an ω -tamed almost complex structure on X preserving TZ , i.e. making Z an almost complex submanifold. The relative Gromov-Witten invariants are defined by counting the number of stable J -holomorphic maps intersecting Z at finitely many points with prescribed tangency. More precisely, fix a k -tuple $T_k = (t_1, \dots, t_k)$ of positive integers, consider a marked pre-stable curve

$$(C, x_1, \dots, x_m, y_1, \dots, y_k)$$

and stable J -holomorphic map $f : C \rightarrow X$ such that the divisor f^*Z is

$$f^*Z = \sum_i t_i y_i.$$

One would like to consider the moduli space of such curves and apply the virtual neighborhood technique to construct the relative invariants. But this scheme needs modification as the moduli space is not compact. It is true that for a sequence of J -holomorphic maps (Σ_n, f_n) as above, by possibly passing to a subsequence, f_n will still converge to a stable J -holomorphic map (Σ, f) . However the limit (Σ, f) may have some Z -components, i.e. components whose images under f lie entirely in Z .

To deal with this problem the authors in [LR] adopt the open cylinder model. Choose a Hamiltonian S^1 function H in a closed ϵ -symplectic tubular neighborhood X_0 of Z with $H(X_0) = [-\epsilon, 0]$ and $Z = H^{-1}(-\epsilon)$. Next we need to choose an almost complex structure with nice properties near Z . An almost complex structure J on X is said to be tamed relative to Z if J is ω -tamed, S^1 -invariant for some (X_0, H) , and such that Z is an almost complex submanifold. The set of such J is nonempty and forms a contractible space. With such a choice of almost complex structure, X_0 can be viewed as a neighborhood of the zero section of the complex line bundle $N_{Z|X}$ with the S^1 action given by the complex multiplication $e^{2\pi i\theta}$. Now we remove Z . The end of $X - Z$ is simply $X_0 - Z$. Recall that the punctured disc $D - \{0\}$ is biholomorphic to the half cylinder $S^1 \times [0, \infty)$. Therefore, as an almost complex manifold, $X_0 - Z$ can be viewed as the translation invariant almost complex half cylinder $P \times [0, \infty)$ where $P = H^{-1}(0)$. In this sense, $X - Z$ is viewed as a manifold with almost complex cylinder end.

Now we consider a holomorphic map in the cylinder model where the marked points mapped into Z are removed from the domain surface. Again we can view a punctured neighborhood of each of these marked points as a half cylinder $S^1 \times [0, \infty)$. With such a J , a J -holomorphic map of X intersecting Z at finitely many points then exactly corresponds to a J -holomorphic map to the open manifold $X - Z$ from a punctured Riemann surface which converges to (a multiple of) an S^1 -orbit at a puncture point.

Now we reconsider the convergence of (Σ_n, f_n) in the cylinder model. The creation of a Z -component f_ν corresponds to disappearance of a part of $\text{im}(f_n)$ to infinity. We can use translation to rescale back the missing part of $\text{im}(f_n)$. In the limit, we may obtain a stable map \tilde{f}_ν into $P \times \mathbb{R}$. When we obtain X from the cylinder model, we need to collapse the S^1 -action at infinity. Therefore, in the limit, we need to take into account maps into the closure of $P \times \mathbb{R}$. Let Y be the projective completion of the normal bundle $N_{Z|X}$, i.e. $Y = \mathbb{P}(N_{Z|X} \oplus \mathbb{C})$. Then Y has a zero section Z_0 and an infinity section Z_∞ . We view $Z \subset X$ as the zero section. One can further show that \tilde{f}_ν is indeed a stable map into Y with the stability specified below.

To form a compact moduli space of such maps we thus must allow the target X to degenerate as well (compare with [Li1]). For any non-negative integer m , construct Y_m by gluing together m copies of Y , where the infinity section of the i^{th} component is glued to the zero section of the $(i+1)^{\text{st}}$ component for $1 \leq i \leq m$. Denote the zero section of the i^{th} component by Z_{i-1} , and the infinity section by Z_i , so $\text{Sing } Y_m = \cup_{i=1}^{m-1} Z_i$. We will also denote Z_m by Z_∞ if there is no possible confusion. Define X_m by gluing X along Z to Y_m along Z_0 . Thus $\text{Sing } X_m = \cup_{i=0}^{m-1} Z_i$ and $X_0 = X$. $X_0 = X$ will be referred to as the root component and the other irreducible components will be called the bubble components. Let $\text{Aut}_Z Y_m$ be the group of automorphisms of Y_m preserving Z_0, Z_m , and the morphism to Z . And let $\text{Aut}_Z X_m$ be the group of automorphisms of X_m preserving X (and Z) and with restriction to Y_m being contained in $\text{Aut}_Z Y_m$ (so $\text{Aut}_Z X_m = \text{Aut}_Z Y_m \cong (\mathbb{C}^*)^m$, where each factor of $(\mathbb{C}^*)^m$ dilates the fibers of the \mathbb{P}^1 -bundle $Y_i \rightarrow Z_i$). Denote by $\pi[m] : X_m \rightarrow X$ the map which is the identity on the root component X_0 and contracts all the bubble components to Z_0 via the fiber bundle projections.

Now consider a nodal curve C mapped into X_m by $f : C \rightarrow X_m$ with specified tangency to Z . There are two types of marked points:

- (i) absolute marked points whose images under f lie outside Z , labeled by x_i ,
- (ii) relative marked points which are mapped into Z by f , labeled by y_j .

A relative J -holomorphic map $f : C \rightarrow X_m$ is said to be pre-deformable if $f^{-1}(Z_i)$ consists of a union of nodes such that for each node $p \in f^{-1}(Z_i)$, $i = 1, 2, \dots, m$, the two branches at the node are mapped to different irreducible components of X_m and the orders of contact to Z_i are equal.

An isomorphism of two such J -holomorphic maps f and f' to X_m consists of a diagram

$$\begin{array}{ccc}
 (C, x_1, \dots, x_l, y_1, \dots, y_k) & \xrightarrow{f} & X_m \\
 h \downarrow & & \downarrow t \\
 (C', x'_1, \dots, x'_l, y'_1, \dots, y'_k) & \xrightarrow{f'} & X_m
 \end{array}$$

where h is an isomorphism of marked curves and $t \in \text{Aut}_Z(X_m)$. With the preceding understood, a relative J -holomorphic map to X_m is said to be stable if it has only finitely many automorphisms.

We introduced the notion of a weighted graph in Remark 2.7. We need to refine it for relative stable maps to (X, Z) . A (connected) relative graph Γ consists of the following data:

- (1) a vertex decorated by $A \in H_2(X; \mathbb{Z})$ and genus g ,
- (2) a tail for each absolute marked point,
- (3) a relative tail for each relative marked point.

Definition 2.8. *Let Γ be a relative graph with k (ordered) relative tails and $T_k = (t_1, \dots, t_k)$, a k -tuple of positive integers forming a partition of $A \cdot Z$. A relative J -holomorphic map to (X, Z) with type (Γ, T_k) consists of a marked curve $(C, x_1, \dots, x_l, y_1, \dots, y_k)$ and a map $f : C \rightarrow X_m$ for some non-negative integer m such that*

- (i) C is a connected curve (possibly reducible) of arithmetic genus g ,
- (ii) the map

$$\pi_m \circ f : C \rightarrow X_m \rightarrow X$$

satisfies $(\pi_m \circ f)_*[C] = A$,

- (iii) the $x_i, 1 \leq i \leq l$, are the absolute marked points,
- (iv) the $y_i, 1 \leq i \leq k$, are the relative marked points,
- (v) $f^*Z_m = \sum_{i=1}^k t_i y_i$.

Let $\overline{\mathcal{M}}_{\Gamma, T_k}(X, Z, J)$ be the moduli space of pre-deformable relative stable J -holomorphic maps with type (Γ, T_k) . Notice that for an element $f : C \rightarrow X_m$ in $\overline{\mathcal{M}}_{\Gamma, T_k}(X, Z, J)$ the intersection pattern with Z_0, \dots, Z_{m-1} is only constrained by the genus condition and the pre-deformability condition.

Now we apply the virtual neighborhood technique to construct $U_{\mathcal{S}_e}^{X, Z}$, $\mathcal{E}_{X, Z}$, $\mathcal{S}_{X, Z}$ as in section 2.1. Consider the configuration space $\overline{\mathcal{B}}_{\Gamma}(X, Z, J)$ of equivalence classes of smooth pre-deformable relative stable maps. Here we still take the equivalence class under \mathbb{C}^* -action on the fibers of $\mathbb{P}(N_{Z|X} \oplus \mathbb{C})$. In particular, the subgroup of \mathbb{C}^* fixing such a map is required to be finite. The maps are required to intersect the Z_i only at finitely many points in the domain curve. Further, at these points, the map is required to have a holomorphic leading term in the normal Taylor expansion for any local chart of X taking Z to a coordinate hyperplane and being holomorphic in the normal direction along Z . Thus the notion of contact order still makes sense, and we can still impose the pre-deformability condition and contact order condition at the y_i being governed by T_k .

Next, we can define Ω^1 similarly. We also need to understand the Obs space. The discussion is similar to that of stable maps.

According to its label, a relative stable map is naturally divided into components of two types:

- (i) a stable map in X intersecting Z transversely, called a rigid factor;

(ii) a stable map in $\mathbb{P}(N_{Z|X} \oplus \mathbb{C})$ such that its projection to Z is stable and it intersects Z_0, Z_∞ transversely, called a rubber factor.

For each component (Σ, f) , we also have the extra condition that f intersects Z_0 or Z_∞ with an order fixed by the graph.

Suppose that $f(y_i) \in Z_0$ or Z_∞ with order t_i . The analog of (1) is

$$L_{\Sigma, f}^{X, Z} : \Omega_r^0 \longrightarrow \Omega_r^{0,1}.$$

Here an element $u \in \Omega_r^0$ is an element of $\Omega^0(f^*TX)$ or $\Omega^0(f^*T\mathbb{P}(N_{Z|X} \oplus \mathbb{C}))$ with **the following property**: Choose a unitary connection on N so that we can decompose the tangent bundle of $\mathbb{P}(N_{Z|X} \oplus \mathbb{C})$ into tangent and normal directions. Near $f(y_i)$, $u(y_i)$ can be decomposed into (u_Z, u_N) where u_Z, u_N are tangent and normal components, respectively. Now we require that u_N vanish at y_i with order t_i . When Σ consists of two components joined at one point, we require their u_Z components be the same at the intersection point.

We can also consider the analog of $\tilde{L}_{\Sigma, f}$,

$$\tilde{L}_{\Sigma, f}^{X, Z} \oplus \oplus_i T_{t_i} : \tilde{\Omega}_r^0 \longrightarrow \tilde{\Omega}_r^{0,1} \oplus \oplus_i \mathcal{J}_i^{t_i}.$$

Here $\tilde{\Omega}_r^0 = \{u \in \Omega^0 \mid u_N(y_i) = 0\}$. Also an element $v \in \tilde{\Omega}_r^{0,1}$ is required to have a simple pole at each y_i such that $\text{Res}_{y_i} v \in TZ$, and if two components are joined together, we require that the sum of the residues be zero. Each summand

$$\mathcal{J}_i^{t_i} \cong \oplus_{j=1}^{t_i-1} \text{Hom}((T_{y_i}\Sigma)^j, N_{f(y_i)})$$

is the $(t_i - 1)$ -jet space, and the map $T_{t_i}(f)$ is the $(t_i - 1)$ -jet of f at y_i , i.e. the first $(t_i - 1)$ terms of the Taylor polynomial.

It is clear that $\text{Coker } \tilde{L}_{\Sigma}^{X, Z}$ has a similar description as $\text{Coker } \tilde{L}_{\Sigma}^X$, the only difference being that we require the residue at each nodal point be in TZ . Moreover,

$$\tilde{L}_{\Sigma, f}^{X, Z} \oplus \oplus_i T_{t_i} = \tilde{L}_{\Sigma, f}^{X, Z} \oplus \oplus_i T_{t_i} \mid_{\text{Ker } \tilde{L}_{\Sigma, f}^{X, Z}}.$$

Finally the process of adding the deformation of a nodal Riemann surface is identical.

In addition to the evaluation maps on $\overline{\mathcal{B}}_{\Gamma, T_k}(X, Z, J)$,

$$\begin{aligned} ev_i^X : \overline{\mathcal{B}}_{\Gamma, T_k}(X, Z, J) &\longrightarrow X, & 1 \leq i \leq l, \\ (\Sigma, x_1, \dots, x_l, y_1, \dots, y_k, f) &\longrightarrow f(x_i), \end{aligned}$$

there are also the evaluations maps

$$\begin{aligned} ev_j^Z : \overline{\mathcal{B}}_{\Gamma, T_k}(X, Z, J) &\longrightarrow Z, & 1 \leq j \leq k, \\ (\Sigma, x_1, \dots, x_l, y_1, \dots, y_k, f) &\longrightarrow f(y_j), \end{aligned}$$

where $Z = Z_m$ if the target of f is X_m .

Definition 2.9. Let $\alpha_i \in H^*(X; \mathbb{R}), 1 \leq i \leq l$, $\beta_j \in H^*(Z; \mathbb{R}), 1 \leq j \leq k$. Define the relative Gromov-Witten invariant

$$\langle \Pi_i \tau_{d_i} \alpha_i \mid \Pi_j \beta_j \rangle_{\Gamma, T_k}^{X, Z} = \frac{1}{|Aut(T_k)|} \int_{U_{S_e}^{X, Z}} S_{X, Z}^* \Theta \wedge \Pi_i \psi_i^{d_i} \wedge (ev_i^X)^* \alpha_i \wedge \Pi_j (ev_j^Z)^* \beta_j,$$

where Θ is the Thom class of the bundle $\mathcal{E}_{X, Z}$ and $Aut(T_k)$ is the symmetry group of the partition T_k . Denote by $\mathcal{T}_k = \{(t_j, \beta_j) \mid j = 1, \dots, k\}$ the weighted partition of $A \cdot Z$. If the vertex of Γ is decorated by (g, A) , we will sometimes write

$$\langle \Pi_i \tau_{d_i} \alpha_i \mid \mathcal{T}_k \rangle_{g, A}^{X, Z}$$

for $\langle \Pi_i \tau_{d_i} \alpha_i \mid \Pi_j \beta_j \rangle_{\Gamma, T_k}^{X, Z}$.

Remark 2.10. In [LR] only invariants without descendent classes were considered. But it is straightforward to extend the definition of [LR] to include descendent classes.

We can decorate the tail of a relative graph Γ by (d_i, α_i) as in the absolute case. We can further decorate the relative tails by the weighted partition \mathcal{T}_k . Denote the resulting weighted relative graph by $\Gamma\{(d_i, \alpha_i)\}|\mathcal{T}_k$. In [LR] the source curve is required to be connected. We will also need to use a disconnected version. For a disjoint union Γ^\bullet of weighted relative graphs and a corresponding disjoint union of partitions, still denoted by \mathcal{T}_k , we use $\langle \Gamma^\bullet\{(d_i, \alpha_i)\}|\mathcal{T}_k \rangle^{X, Z}$ to denote the corresponding relative invariants with a disconnected domain, which is simply the product of the connected relative invariants. Notice that although we use \bullet in our notation following [MP], our disconnected invariants are different. The disconnected invariants there depend only on the genus, while ours depend on the finer graph data.

2.3. Partial orderings on relative GW invariants. In [MP], the authors first introduced a partial order on the set of relative Gromov-Witten invariants of a \mathbb{P}^1 -bundle. The authors, [HLR], refined their partial order on the set of relative Gromov-Witten invariants of a Blow-up manifold relative to the exceptional divisor, and used this partial order to obtain a Blow-up correspondence of absolute/relative Gromov-Witten theory. In this subsection, we will review the partial order on the set of relative Gromov-Witten invariants.

First of all, all Gromov-Witten invariants vanish if $A \in H_2(X, \mathbb{Z})$ is not an effective curve class. We define a partial ordering on $H_2(X, \mathbb{Z})$ as follows:

$$A' < A$$

if $A - A'$ is a nonzero effective curve class.

The set of pairs (m, δ) where $m \in \mathbb{Z}_{>0}$ and $\delta \in H^*(Z, \mathbb{Q})$ is partially ordered by the following size relation

$$(2) \quad (m, \delta) > (m', \delta')$$

if $m > m'$ or if $m = m'$ and $\deg(\delta) > \deg(\delta')$.

Let μ be a partition weighted by the cohomology of Z , i.e.,

$$\mu = \{(\mu_1, \delta_{r_1}), \dots, (\mu_{\ell(\mu)}, \delta_{r_{\ell(\mu)}})\}.$$

We may place the pairs of μ in decreasing order by size (2). We define

$$\deg(\mu) = \sum \deg(\delta_{r_i}).$$

A lexicographic ordering on weighted partitions is defined as follows:

$$\mu \overset{l}{>} \mu'$$

if, after placing the pairs in μ and μ' in decreasing order by size, the first pair for which μ and μ' differ in size is larger for μ .

For the nondescendent relative Gromov-Witten invariant

$$\langle \varpi \mid \mu \rangle_{g,A}^{X,Z},$$

denote by $\|\varpi\|$ the number of absolute insertions.

Definition 2.11. A partial ordering $\overset{\circ}{<}$ on the set of nondescendent relative Gromov-Witten invariants is defined as follows:

$$\langle \varpi' \mid \mu' \rangle_{g',A'}^{X,Z} \overset{\circ}{<} \langle \varpi \mid \mu \rangle_{g,A}^{X,Z}$$

if one of the conditions below holds

- (a) $A' < A$,
- (b) equality in (a) and $g' < g$,
- (c) equality in (a)-(b) and $\|\varpi'\| < \|\varpi\|$,
- (d) equality in (a)-(c) and $\deg(\mu') > \deg(\mu)$,
- (e) equality in (a)-(d) and $\mu' \overset{l}{>} \mu$.

2.4. Degeneration formula. Now we describe the degeneration formula of GW-invariants under symplectic cutting.

As an operation on topological spaces, the symplectic cut is essentially collapsing the circle orbits in the hypersurface $H^{-1}(0)$ to points in Z .

Suppose that $X_0 \subset X$ is an open codimension zero submanifold with a Hamiltonian S^1 -action. Let $H : X_0 \rightarrow \mathbb{R}$ be a Hamiltonian function with 0 as a regular value. If $H^{-1}(0)$ is a separating hypersurface of X_0 , then we obtain two connected manifolds X_0^\pm with boundary $\partial X_0^\pm = H^{-1}(0)$. Suppose further that S^1 acts freely on $H^{-1}(0)$. Then the symplectic reduction $Z = H^{-1}(0)/S^1$ is canonically a symplectic manifold of dimension 2 less. Collapsing the S^1 -action on $\partial X^\pm = H^{-1}(0)$, we obtain closed smooth manifolds \overline{X}_0^\pm containing respectively real codimension 2 submanifolds $Z^\pm = Z$ with opposite normal bundles. Furthermore \overline{X}_0^\pm admits a symplectic structure $\overline{\omega}^\pm$ which agrees with the restriction of ω away from Z , and whose restriction to Z^\pm agrees with the canonical symplectic structure ω_Z on Z from symplectic reduction.

This is neatly shown by considering $X_0 \times \mathbb{C}$ equipped with appropriate product symplectic structures and the product S^1 -action on $X_0 \times \mathbb{C}$, where

S^1 acts on \mathbb{C} by complex multiplication. The extended action is Hamiltonian if we use the standard symplectic structure $\sqrt{-1}dw \wedge d\bar{w}$ or its negative on the \mathbb{C} factor. Then the moment map is

$$\mu_+(u, w) = H(u) + |w|^2 : X_0 \times \mathbb{C} \rightarrow \mathbb{R},$$

and $\mu_+^{-1}(0)$ is the disjoint union of S^1 -invariant sets

$$\{(u, w) | H(u) = -|w|^2 < 0\} \quad \text{and} \quad \{(u, 0) | H(u) = 0\}.$$

We define \overline{X}_0^+ to be the symplectic reduction $\mu_+^{-1}(0)/S^1$. Then \overline{X}_0^+ is the disjoint union of an open symplectic submanifold and a closed codimension 2 symplectic submanifold identified with (Z, ω_Z) . The open piece can be identified symplectically with

$$X_0^+ = \{u \in X_0 | H(u) < 0\} \subset X_0$$

by the map $u \rightarrow (u, \sqrt{-H(u)})$.

Similarly, if we use $-\mathbf{i}dw \wedge d\bar{w}$, then the moment map is

$$\mu_-(u, w) = H(u) - |w|^2 : X_0 \times \mathbb{C} \rightarrow \mathbb{R}$$

and the corresponding symplectic reduction $\mu_-^{-1}(0)/S^1$, denoted by \overline{X}_0^- , is the disjoint union of an open piece identified symplectically with

$$X_0^- = \{u \in X_0 | H(u) > 0\}$$

by the map $\phi_0^- : u \rightarrow (u, \sqrt{H(u)})$, and a closed codimension 2 symplectic submanifold identified with (Z, ω_Z) .

We finally define \overline{X}^+ and \overline{X}^- . \overline{X}^+ is simply \overline{X}_0^+ , while \overline{X}^- is obtained from gluing symplectically X^- and \overline{X}_0^- along X_0 via ϕ_0^- . Notice that $\overline{X}^- = (X^- - X_0) \cup \overline{X}_0^-$ as a set.

The two symplectic manifolds $(\overline{X}^\pm, \overline{\omega}^\pm)$ are called the symplectic cuts of X along $H^{-1}(0)$.

Thus we have a continuous map

$$\pi : X \rightarrow \overline{X}^+ \cup_Z \overline{X}^-.$$

As for the symplectic forms, we have $\omega^+|_Z = \omega^-|_Z$. Hence, the pair (ω^+, ω^-) defines a cohomology class of $\overline{X}^+ \cup_Z \overline{X}^-$, denoted by $[\omega^+ \cup_Z \omega^-]$. It is easy to observe that

$$(3) \quad \pi^*([\omega^+ \cup_Z \omega^-]) = [\omega].$$

Let $B \in H_2(X; \mathbb{Z})$ be in the kernel of

$$\pi_* : H_2(X; \mathbb{Z}) \longrightarrow H_2(\overline{X}^+ \cup_Z \overline{X}^-; \mathbb{Z}).$$

By (3) we have $\omega(B) = 0$. Such a class is called a vanishing cycle. For $A \in H_2(X; \mathbb{Z})$ define $[A] = A + \text{Ker}(\pi_*)$ and

$$(4) \quad \langle \Pi_i \tau_{d_i} \alpha_i \rangle_{g, [A]}^X = \sum_{B \in [A]} \langle \Pi_i \tau_{d_i} \alpha_i \rangle_{g, B}^X.$$

Notice that ω has constant pairing with any element in $[A]$. It follows from the Gromov compactness theorem that there are only finitely many such elements in $[A]$ represented by J -holomorphic stable maps. Therefore, the summation in (4) is finite.

The degeneration formula expresses $\langle \Pi_i \tau_{d_i} \alpha_i \rangle_{g,[A]}^X$ in terms of relative invariants of (\overline{X}^+, Z) and (\overline{X}^-, Z) possibly with disconnected domains.

To begin with, we need to assume that the cohomology class α_i is of the form

$$\alpha_i = \pi^*(\alpha^+ \cup_Z \alpha^-).$$

Here $\alpha_i^\pm \in H^*(\overline{X}^\pm; \mathbb{R})$ are classes with $\alpha_i^+|_Z = \alpha_i^-|_Z$ so that they give rise to a class $\alpha_i^+ \cup_Z \alpha_i^- \in H^*(\overline{X}^+ \cup_Z \overline{X}^-; \mathbb{R})$.

Next, we proceed to write down the degeneration formula. We first specify the relevant topological type of a marked Riemann surface mapped into $\overline{X}^+ \cup_Z \overline{X}^-$ with the following properties:

- (i) Each connected component is mapped either into \overline{X}^+ or \overline{X}^- and carries a respective degree 2 homology class;
- (ii) The images of two distinct connected components only intersect each other along Z ;
- (iii) No two connected components which are both mapped into \overline{X}^+ or \overline{X}^- intersect each other;
- (iv) The marked points are not mapped to Z ;
- (v) Each point in the domain mapped to Z carries a positive integer (representing the order of tangency).

By abuse of language we call the above data a $(\overline{X}^+, \overline{X}^-)$ -graph. Such a graph gives rise to two relative graphs of (\overline{X}^+, Z) and (\overline{X}^-, Z) from (i-iv), each possibly being disconnected. We denote them by Γ_+^\bullet and Γ_-^\bullet respectively. From (v) we also get two partitions T_+ and T_- . We call a $(\overline{X}^+, \overline{X}^-)$ -graph a degenerate (g, A, l) -graph if the resulting pairs (Γ_+^\bullet, T_+) and (Γ_-^\bullet, T_-) satisfy the following constraints: the total number of marked points is l , the relative tails are the same, i.e. $T_+ = T_-$, and the identification of relative tails produces a connected graph of X with total homology class $\pi_*[A]$ and arithmetic genus g .

Let $\{\beta_a\}$ be a self-dual basis of $H^*(Z; \mathbb{R})$ and $\eta^{ab} = \int_Z \beta_a \cup \beta_b$. Given g, A and l , consider a degenerate (g, A, l) -graph. Let $T_k = T_+ = T_-$ and \mathcal{T}_k be a weighted partition $\{t_j, \beta_{a_j}\}$. Let $\mathcal{T}'_k = \{t_j, \beta_{a_{j'}}\}$ be the dual weighted partition.

The degeneration formula for $\langle \Pi_i \tau_{d_i} \alpha_i \rangle_{g,[A]}^X$ then reads as follows,

$$\begin{aligned} & \langle \Pi_i \tau_{d_i} \alpha_i \rangle_{g,[A]}^X \\ &= \sum \langle \Gamma^\bullet \{(d_i, \alpha_i^+)\} | \mathcal{T}_k \rangle_{\overline{X}^+, Z} \Delta(\mathcal{T}_k) \langle \Gamma^\bullet \{(d_i, \alpha_i^-)\} | \mathcal{T}'_k \rangle_{\overline{X}^-, Z}, \end{aligned}$$

where the summation is taken over all degenerate (g, A, l) -graphs, and

$$\Delta(\mathcal{T}_k) = \prod_j t_j |\text{Aut}(T_k)|.$$

3. RELATIVE GW-INVARIANTS OF \mathbb{P}^1 -BUNDLES

Suppose that Z is a symplectic submanifold of X of codimension 2. When applying the degeneration formula, we often need to express the absolute Gromov-Witten invariants of X as a summation of products of relative Gromov-Witten invariants of symplectic cuts of X . Thus if we want to obtain a comparison theorem of Gromov-Witten invariant by the degeneration formula, the point will be how to compute the relative Gromov-Witten invariants of a \mathbb{P}^1 -bundle. In this section, we will prove a vanishing theorem for genus zero relative Gromov-Witten invariants of the \mathbb{P}^1 -bundle Y relative to the infinity section and compute some genus zero two-point relative fiber class GW invariants of the \mathbb{P}^1 -bundle Y .

Suppose that L is a line bundle over Z and $Y = \mathbb{P}(L \oplus \mathbb{C})$. Let D, Z be the infinity section and zero section of Y respectively. Let $\beta_1, \dots, \beta_{m_Z}$ be a self-dual basis of $H^*(Z, \mathbb{Q})$ containing the identity element. We will often denote the identity by β_{id} . The degree of β_i is the real grading in $H^*(Z, \mathbb{Q})$. We view β_i as an element of $H^*(Y, \mathbb{Q})$ via the pull-back by $\pi : Y = \mathbb{P}(L \oplus \mathbb{C}) \longrightarrow Z$. Let $[Z], [D] \in H^2(Y, \mathbb{Q})$ denote the cohomology classes associated to the divisors. Define classes in $H^*(Y, \mathbb{Q})$ by

$$\begin{aligned} \gamma_i &= \beta_i, \\ \gamma_{m_Z+i} &= \beta_i \cdot [Z], \\ \gamma_{2m_Z+i} &= \beta_i \cdot [D]. \end{aligned}$$

We will use the following notation:

$$\begin{aligned} \gamma_i^\beta &= \beta_i \bmod m_Z, \\ \gamma_i^D &= 1, [Z], \text{ or } [D]. \end{aligned}$$

The second assignment depends upon the integer part of $(i-1)/m_Z$. The set $\{\gamma_1, \dots, \gamma_{2m_Z}\}$ determines a basis of $H^*(Y, \mathbb{Q})$.

Since $D \cong Z$ as topological manifolds, $H^*(D, \mathbb{Q})$ is isomorphic to $H^*(Z, \mathbb{Q})$. If there is no confusion, we will also use $\beta_1, \dots, \beta_{m_Z}$ as a self-dual basis of $H^*(D, \mathbb{Q})$.

3.1. Fiber class invariants. In this subsection, we mainly compute some genus zero relative GW invariants of \mathbb{P}^1 -bundles with a fiber class. According to [MP], we may transfer the computation of this invariant on the \mathbb{P}^1 -bundle into that of some associated invariants on \mathbb{P}^1 .

Suppose that L is a line bundle over Z and

$$Y = \mathbb{P}(L \oplus \mathcal{O}) = \{(z, l) | z \in Z, l \subset L_z \oplus \mathbb{C}\},$$

where L_z is the fiber of L at z .

Denote by $\pi : Y \longrightarrow Z$ the projection of the \mathbb{P}^1 -bundle Y . Moreover there are two inclusions (sections) of Z in $Y = \mathbb{P}(L \oplus \mathcal{O})$:

- (1) the “zero section” $z \longrightarrow (z, 0 \oplus \mathbf{C})$, denoted by Z ,
- (2) the “section at infinity” $z \longrightarrow (z, L_z \oplus 0)$, denoted by D .

Let Γ be the relative graph with the following data

- (1) a vertex decorated by $A = sF \in H_2(Y, \mathbb{Z})$ and genus zero;
- (2) k relative tails;
- (3) l absolute tails.

For any non-negative integer m , define Y_m by gluing together m copies of Y , where the infinity section of the i^{th} component is glued to the zero section of the $(i+1)^{st}$ ($1 \leq i \leq m$) component; see Section 2.2 for details. Denote by $\pi[m] : Y_m \longrightarrow Y$ the map which is the identity on the root component Y_0 and contract all the bubble components to D_0 via the projection of the fiber bundle of Y_i .

Let $T_k = \{t_1, \dots, t_k\}$ be a k -tuple of positive integers forming a partition of s . Denote by $\overline{\mathcal{M}}_{\Gamma, T_k}(Y, D)$ the moduli space of morphisms

$$f : (C, x_0, \dots, x_l; y_1, \dots, y_k) \longrightarrow (Y_m, D_\infty),$$

such that

- (1) $(C, x_0, \dots, x_l; y_1, \dots, y_k)$ is a prestable curve of genus zero with l absolute marked points x_0, \dots, x_l and k relative marked points y_1, \dots, y_k .
- (2) $f^{-1}(D_\infty) = \sum t_i y_i$ as Cartier divisor and $\deg(\pi[m] \circ f) = s$.
- (3) The predeformability condition: The preimage of the singular locus $\text{Sing } Y_m = \cup_{i=0}^{m-1} D_i$ of Y_m is a union of nodes of C , and if p is one such node, then the two branches of C at p map into different irreducible components of Y_m , and their orders of contact with the divisor D_i (in their respective components of Y_m) are equal. The morphism f is also required to satisfy a stability condition that there are no infinitesimal automorphisms of the sequence of maps $(C, x_0, \dots, x_l; y_1, \dots, y_k) \longrightarrow Y_m \xrightarrow{\pi[m]} Y$ where the allowed automorphisms of the map from Y_m to Y are $\text{Aut}_D(Y_m)$.
- (4) The automorphism group of f is finite.

Two such morphisms are isomorphic if they differ by an isomorphism of the domain and an automorphism of (Y_m, D_0, D_∞) . In particular, this defines the automorphism group in the stability condition (4) above.

We introduce some notations which are used in [Li1]. For any non-negative integer m , let

$$\mathbb{P}^1[m] = \mathbb{P}_{(0)}^1 \cup \mathbb{P}_{(1)}^1 \cup \dots \cup \mathbb{P}_{(m)}^1$$

be a chain of $m+1$ copies \mathbb{P}^1 , where $\mathbb{P}_{(l)}^1$ is glued to $\mathbb{P}_{(l+1)}^1$ at $p_1^{(l)}$ for $0 \leq l \leq m-1$. The irreducible component $\mathbb{P}_{(0)}^1$ will also be referred to as the root component and the other irreducible components will be called

the bubble components. A point $p_1^{(m)} \neq p_1^{(m-1)}$ is fixed on $\mathbb{P}_{(m)}^1$. Denote still by $\pi[m] : \mathbb{P}^1[m] \longrightarrow \mathbb{P}^1$ the map which is the identity on the root component and contracts all the bubble components to $p_1^{(0)}$. For $m > 0$, let

$$\mathbb{P}^1(m) = \mathbb{P}_{(1)}^1 \cup \cdots \cup \mathbb{P}_{(m)}^1$$

denote the union of bubble components of $\mathbb{P}^1[m]$.

Similar to the case of Y_m , we may define the associated moduli space $\overline{\mathcal{M}}_\Gamma(\mathbb{P}^1, p_1^{(0)}; T_1)$ of relative stable maps to $(\mathbb{P}^1, p_1^{(0)})$, see [Li1] for its definition.

Next, we first review Maulik-Pandharipande's algorithm [MP] which reduces the relative Gromov-Witten invariant of (Y, D) of fiber class to that of (\mathbb{P}^1, p_1) . Note that the moduli space of stable relative maps

$$\overline{\mathcal{M}}_Y = \overline{\mathcal{M}}_{\Gamma, T_k}(Y, D)$$

is fibered over Z ,

$$(5) \quad \pi : \overline{\mathcal{M}}_Y \longrightarrow Z,$$

with fiber isomorphic to the moduli space of maps of degree s to \mathbb{P}^1 relative to the infinity point p_1 with tangency order s :

$$\overline{\mathcal{M}}_{\mathbb{P}^1} = \overline{\mathcal{M}}_{\Gamma, T_k}(\mathbb{P}^1, p_1).$$

In fact, $\overline{\mathcal{M}}_Y$ is the fiber bundle constructed from the principal S^1 -bundle associated to L and a standard S^1 -action on $\overline{\mathcal{M}}_{\mathbb{P}^1}$.

The π -relative obstruction theory of $\overline{\mathcal{M}}_Y$ is obtained from the $\overline{\mathcal{M}}_{\mathbb{P}^1}$ -fiber bundle structure over Z . The relationship between the π -relative virtual fundamental class $[\overline{\mathcal{M}}_{\overline{M}+}]^{vir_\pi}$ and the virtual fundamental class $[\overline{\mathcal{M}}_{\overline{M}+}]^{vir}$ is given by the equation

$$(6) \quad [\overline{\mathcal{M}}_Y]^{vir} = c_{top}(\mathbb{E} \otimes TZ) \cap [\overline{\mathcal{M}}_Y]^{vir_\pi}$$

where \mathbb{E} is the Hodge bundle. Since we only consider the case of genus zero, (6) can be written as

$$[\overline{\mathcal{M}}_Y]^{vir} = [\overline{\mathcal{M}}_Y]^{vir_\pi}.$$

By integrating along the fiber, we can compute the relative Gromov-Witten invariants of Y by computing the equivariant integrations in the relative Gromov-Witten theory of \mathbb{P}^1 ; see [MP] for the details.

Let $\mathcal{T}_k = \{(t_i, \beta_i)\}$ be the cohomology weighted partition of s . By definition, we have

$$(7) \quad \begin{aligned} \langle \tau_{d_1-1}\gamma_1, \dots, \tau_{d_l-1}\gamma_l \mid \mathcal{T}_k \rangle_\Gamma^{Y,D} &= \int_{[\overline{\mathcal{M}}_Y]^{vir}} \prod_{i=1}^l \psi_i^{d_i-1} ev_i^* \gamma_i \wedge \prod_j ev_j^* \beta_j \\ &= \frac{1}{|\text{Aut}(\mathcal{T}_k)|} \int_Z \left(\prod_i \gamma_i^\delta \prod_j \beta_j \cap \pi_* \left(\prod_i \psi_i^{d_i-1} ev_i^* (\gamma_i^{D_0}) \cap [\overline{\mathcal{M}}_Y]^{vir_\pi} \right) \right), \end{aligned}$$

where the interior push-forward

$$\pi_*\left(\prod_i \psi_i^{d_i-1} ev_i^*(\gamma_i^{D_0}) \cap [M_Y]^{vir\pi}\right)$$

is obtained from the corresponding Hodge integral in the equivariant Gromov-Witten theory of (\mathbb{P}^1, p_1) after replacing the hyperplane class on \mathbb{CP}^∞ by $C_1(L)$.

Therefore, via (7), we may reduce the computation of relative Gromov-Witten invariants $\langle \tau_{d_1-1}\gamma_1, \dots, \tau_{d_l-1}\gamma_l \mid \mathcal{T}_k \rangle_{\Gamma}^{Y,D}$ to that of

$$\langle \tau_{d_1-1}\delta_1, \dots, \tau_{d_l-1}\delta_l \mid pt, \dots, pt \rangle_{\Gamma, \mathcal{T}_k}^{\mathbb{P}^1, p_1},$$

where $\delta_i \in H^*(\mathbb{P}^1, \mathbb{Q})$, $1 \leq i \leq l$.

About the two point genus zero relative Gromov-Witten invariant of (\mathbb{P}^1, p_1) , we have

Lemma 3.1. *Let $\varpi \in H^2(\mathbb{P}^1, \mathbb{Q})$.*

- (i) *If $d \neq s$, then $\langle \tau_{d-1}\varpi \mid (s, [pt]) \rangle_s^{\mathbb{P}^1, p_1} = 0$.*
- (ii) *For $s > 0$, we have*

$$\langle \tau_{s-1}\varpi \mid (s, [pt]) \rangle_s^{\mathbb{P}^1, p_1} = \frac{1}{s!}.$$

The proof of (i) follows from a simple dimension calculation and (ii) of the lemma is Lemma 1.4 of [OP]. In [HLR], the authors generalized the result to general projective space \mathbb{P}^n via localization techniques.

Proposition 3.2. *Let $s > 0$.*

- (i) *Let $\mathcal{T}_k = \{(t_i, \beta_i)\}$ be a cohomology weighted partition of s . Then*

$$\langle \pi^*\alpha_1, \dots, \pi^*\alpha_q, \beta_1 \cdot [Z], \dots, \beta_l \cdot [Z] \mid \mathcal{T}_k \rangle_{sF}^{Y,D} = 0$$

except for $s = k = 1$ and $q = 0$.

- (ii) *For $s > 0$, we have the two-point relative invariant*

$$\langle \tau_{d-1}(\beta_0 \cdot [Z]) \mid (s, \beta_\infty) \rangle_{sF}^{Y,D} = \begin{cases} \frac{1}{s!} \int_Z \beta_0 \wedge \beta_\infty, & d = s \\ 0, & d \neq s \end{cases},$$

where $\beta_0 \in H^(Z, \mathbb{Q})$ and $\beta_\infty \in H^*(D, \mathbb{Q})$.*

- (iii) *For $s = k = 1$, we have*

$$\langle \iota^!(\beta_1), \dots, \iota^!(\beta_l) \mid (1, \gamma) \rangle_F^{Y,D} = \int_Z \beta_1 \wedge \dots \wedge \beta_l \wedge \gamma.$$

Proof. (i). From (7), we are reduced to a relative Gromov-Witten invariant of \mathbb{P}^1 of the form

$$\langle \mathbb{P}^1, \dots, \mathbb{P}^1, [pt], \dots, [pt] \mid (t_1, [pt]), \dots, (t_k, [pt]) \rangle_s^{\mathbb{P}^1, p_1}.$$

A dimension count shows that this invariant of \mathbb{P}^1 is nonzero only if $s + k = 2 - q$. Since $s > 0$ and $k > 0$, the only possibility is $s = k = 1$ and $q = 0$.

The proof of (ii) directly follows from (7) and Lemma 3.1.

(iii). From (7), we have

$$\begin{aligned} & \langle \iota^!(\beta_1), \dots, \iota^!(\beta_l) \mid (1, \gamma) \rangle_F^{Y,D} \\ &= \int_Z \beta_1 \wedge \dots \wedge \beta_l \wedge \gamma \langle [pt], \dots, [pt] \mid (1, [pt]) \rangle_1^{\mathbb{P}^1, p_1}. \end{aligned}$$

It remains to prove $\langle [pt], \dots, [pt] \mid (1, [pt]) \rangle_1^{\mathbb{P}^1, p_1} = 1$. In fact, we consider the absolute invariant of \mathbb{P}^1 with $l+1$ point insertions: $\langle [pt], \dots, [pt] \rangle_1^{\mathbb{P}^1}$. First of all, by divisor axiom, we know that this absolute invariant equals 1. We apply the degeneration formula to this invariant of \mathbb{P}^1 and distribute one point insertion to one side and other l point insertions to other side. Then we have

$$\begin{aligned} 1 &= \langle [pt], \dots, [pt] \rangle_1^{\mathbb{P}^1} \\ &= \langle [pt], \dots, [pt] \mid (1, [pt]) \rangle_1^{\mathbb{P}^1, p_1} \langle [pt] \mid (1, [pt]) \rangle_1^{\mathbb{P}^1, p_1} \\ &= \langle [pt], \dots, [pt] \mid (1, [pt]) \rangle_1^{\mathbb{P}^1, p_1}. \end{aligned}$$

In the last equality, we used Lemma 3.1. This proved (iii). \square

3.2. A vanishing theorem. In this subsection, we will prove a vanishing result for some relative Gromov-Witten invariants of \mathbb{P}^1 -bundle, in particular, for some non-fiber homology class invariants.

Let Γ_0 be a relative graph with the following data:

- (i) a vertex decorated by a homology class $A \in H_2(Y, \mathbb{Q})$ and genus zero,
- (ii) $l+q$ tails associated to $l+q$ absolute marked points,
- (iii) k relative tails associated to k relative marked points.

Denote by A the homology class of the relative stable map (Σ, f) to (Y, D) and by F the homology class of a fiber of Y . Let $T_k = \{t_1, \dots, t_k\}$ be a partition of $D \cdot A$ and $d_i, 1 \leq i \leq l$, be positive integers. Denote $d = \sum_{i=1}^l d_i$. Denote by $\iota : Z \rightarrow Y$ the inclusion of Z into Y via the zero section of Y . Then for any $\beta \in H^*(Z, \mathbb{R})$, the inclusion map i pushes forward the class β to a cohomology class $\iota^!(\beta) \in H^*(Y, \mathbb{Q})$, determined by the pull-back map ι^* and Poincaré duality.

Proposition 3.3. *Suppose $A \neq sF$ or $k+l+q \geq 3$. Assume that $Z^*(A) \geq \sum d_i$ and $c_1(L)(C) \geq 0$ for any J -holomorphic curve C into Z . Then for any $\beta_i \in H^*(Z, \mathbb{Q})$, $1 \leq i \leq l$, and any weighted partition $\mathcal{T}_k = \{(t_i, \delta_i)\}$ of $D \cdot A$, we have*

$$\langle \varpi, \tau_{d_1-1} \iota^!(\beta_1), \dots, \tau_{d_l-1} \iota^!(\beta_l) \mid \mathcal{T}_k \rangle_{\Gamma_0, T_k}^{Y,D} = 0,$$

where ϖ consists of insertions of the form $\pi^* \alpha_1, \dots, \pi^* \alpha_q$.

Proof. The projection $\pi : Y = \mathbb{P}(L \oplus \mathbb{C}) \rightarrow Z$ induces a map between the moduli spaces, denoted also by π ,

$$(8) \quad \pi : \overline{\mathcal{B}}_{\Gamma_0, T_k}(Y, D, J) \rightarrow \overline{\mathcal{B}}_{\pi_*(A)}^Z(0, k+l+q, J),$$

where π contracts the unstable rational component whose image is a fiber. π is well-defined if $A \neq sF$ or $k + l + q \geq 3$. By the definition, π commutes with the evaluation map. Furthermore, there is also a natural map (denoted by π as well) on $\Omega^{0,1}$ commuting with the map on configuration spaces. Moreover, $\bar{\partial}$ commutes with π . Hence, it induces a map from $\text{Coker } L^{Y,D}$ to $\text{Coker } L^Z$. We claim that π induces a map on a virtual neighborhood.

Let ω be an integral symplectic form on Z . Using Siebert's construction [S1], we can construct a bundle \mathcal{E} dominating the local obstruction bundle generated by $\text{Coker } L_{\pi(\Sigma,f)}^Z$. $\pi^*\mathcal{E}$ is a bundle over $\bar{\mathcal{B}}_{\Gamma_0,T_k}(Y,D,J)$. We want to show that $\pi^*\mathcal{E}$ dominates its local obstruction bundle. Let (Σ, f) be a relative stable map of (Y, D) . Then we have

Lemma 3.4. *$\text{Coker } L_{\Sigma,f}^{Y,D}$ is isomorphic to $\text{Coker } L_{\pi(\Sigma,f)}^Z$.*

Proof. It is well-known that a stable map can be naturally decomposed into connected components lying outside of D (rigid factors) or completely inside D (rubber factors). Let (Σ, f) be a rigid factor or a rubber factor with relative marked points x_1, \dots, x_r such that $f(x_i) \in Z$ or D with order k_i . In both cases, it is a stable map into Y . We take the complex as

$$\tilde{L}_{\Sigma,f}^{Y,D} \times \sum T_{x_i}^{k_i} : \{u \in \Omega^0(f^*TY) \mid u(x_i) \in TZ\} \longrightarrow \Omega^{0,1}(f^*TY \otimes_i \mathcal{O}_{\Sigma}(x_i)) \oplus_i \mathcal{J}_{x_i}^{k_i}.$$

We first study the cohomology H_L^0, H_L^1 of $\tilde{L}_{\Sigma,f}^{Y,D}$. There is a short exact sequence

$$(9) \quad 0 \longrightarrow V \longrightarrow TY \longrightarrow \pi^*TZ \longrightarrow 0,$$

where V is the vertical tangent bundle. It induces a short exact sequence

$$(10) \quad 0 \longrightarrow f^*V \longrightarrow f^*TY \longrightarrow f^*\pi^*TZ \longrightarrow 0.$$

Choose a Hermitian metric and a unitary connection on L . It induces a splitting of (9). We choose a metric of TY as the direct sum of the metric on V and π^*TZ , where the second one is induced from a metric on Z . The Levi-Civita connection is a direct sum. Then f^*V is a holomorphic line bundle with respect to pullback of the Levi-Civita connection.

(10) induces a long exact sequence in cohomology

$$\begin{aligned} 0 &\longrightarrow H^0(f^*V) \longrightarrow H^0(f^*TY) \longrightarrow H^0(f^*\pi^*TZ) \\ &\longrightarrow H^1(f^*V) \longrightarrow H^1(f^*TY) \longrightarrow H^1(f^*\pi^*TZ) \longrightarrow 0. \end{aligned}$$

It induces exact sequences

$$0 \longrightarrow H_L^0(f^*V) \longrightarrow H_L^0(f^*TY) \longrightarrow H_L^0(f^*\pi^*TZ),$$

$$H_L^1(f^*V) \longrightarrow H_L^1(f^*TY) \longrightarrow H_L^1(f^*\pi^*TZ) \longrightarrow 0.$$

Note that the normal bundle at the zero or infinity section is the restriction of V . It is obvious that $H_L^1(f^*\pi^*TZ) = H^1(f^*\pi^*TZ)$. An element of

$H_L^1(f^*V)$ with residue in TZ must have zero residue. Therefore,

$$(11) \quad H_L^1(f^*V) = H^1(\tilde{f}^*V),$$

where $(\tilde{\Sigma}, \tilde{f})$ is obtained from (Σ, f) by dropping the new marked points. For the same reason,

$$H_L^0(f^*V) = \{v \in H^0(f^*V) \mid v(x_i) = 0\}.$$

We claim that $H^1(\tilde{f}^*V) = 0$. Note that since Σ is a tree of \mathbb{P}^1 's, we see that $H^1(L) = 0$ for any line bundle L on Σ satisfying $\deg(L|_{\bar{\Sigma}}) \geq 0$ for any irreducible component $\bar{\Sigma}$ of $\tilde{\Sigma}$.

Now we have

$$\begin{aligned} \deg(\tilde{f}^*V|_{\bar{\Sigma}}) &= \tilde{f}_*[\tilde{\Sigma}] \cdot c_1(V) = \tilde{f}_*[\bar{\Sigma}] \cdot (\pi^*c_1(L) + 2\xi) \\ &= c_1(N) \cdot (\pi \circ \tilde{f})_*[\tilde{\Sigma}] + 2(\tilde{f}_*[\tilde{\Sigma}] \cdot \xi) \geq 0. \end{aligned}$$

Applying $L = \tilde{f}^*V$, we conclude that $H_L^1(\tilde{f}^*V) = 0$. Next, we show that

$$(12) \quad \oplus_i T_{x_i}^{k_i} : H_L^0(f^*TY) \longrightarrow \oplus_i T_{x_i}^{k_i}$$

is surjective. It is enough to show that the restriction to $H_L^0(f^*V)$ is surjective. Consider the exact sequence

$$0 \longrightarrow \tilde{f}^*V \otimes_i \mathcal{O}(-k_i x_i) \longrightarrow \tilde{f}^*V \longrightarrow \oplus_i \tilde{f}^*V_{k_i x_i} \longrightarrow 0.$$

It induces a long exact sequence

$$H^0(\tilde{f}^*V) \longrightarrow \oplus_i H^0(\tilde{f}^*V_{k_i x_i}) \longrightarrow H^1(\tilde{f}^*V \otimes_i \mathcal{O}(-k_i x_i)).$$

Over each $\bar{\Sigma}$,

$$\begin{aligned} \deg \tilde{f}^*V \otimes_i \mathcal{O}(-k_i x_i) |_{\bar{\Sigma}} &= \tilde{f}_*[\bar{\Sigma}] \cdot (\pi^*c_1(N) + 2\xi) \\ &= c_1(N) \cdot (\pi \circ \tilde{f})_*[\bar{\Sigma}] + 2(\tilde{f}_*[\bar{\Sigma}] \cdot \xi) \geq 0. \end{aligned}$$

Hence, $H^1(\tilde{f}^*V \otimes_i \mathcal{O}(-k_i x_i)) = 0$. This implies that

$$H^0(\tilde{f}^*V) \longrightarrow \oplus_i H^0(\tilde{f}^*V_{k_i x_i})$$

is surjective. Now we go back to f and drop the constant term in $H^0(\tilde{f}^*V_{k_i x_i})$.

(12) becomes

$$(13) \quad H_L^0(f^*V) \longrightarrow \oplus_i T_{x_i}^{k_i},$$

which is obviously surjective. By (11), we have proved that $\text{Coker}(\tilde{L}_{\Sigma, f}^{Y, D} \times \sum T_{x_i}^{k_i})$ is isomorphic to $H^1(f^*\pi^*TZ)$. Then, we argue that $H^1(f^*\pi^*TZ)$ is isomorphic to $H^1(\pi(f)^*TZ)$. This is obvious if $\pi(f)$ contracts an unstable component \mathbb{P}^1 , $\pi \circ f(\mathbb{P}^1) = \text{constant}$ and \mathbb{P}^1 has one or two special points. Moreover, $\pi(\Sigma)$ is obtained by contracting \mathbb{P}^1 . Note that $f^*\pi^*TZ|_{\mathbb{P}^1}$ is trivial.

The space of meromorphic 1-forms on \mathbb{P}^1 with a simple pole at one or two points is zero or 1-dimensional. If \mathbb{P}^1 has only one special point, the residue at the special point has to be zero. We can simply contract this component

and remove the pole at the other component which \mathbb{P}^1 is connected to. If \mathbb{P}^1 has two special points, the residues at the two points have to be the same. Then we can remove this component and the joint residue at the two special points. Then we identify $H^1(f^*\pi^*TZ)$ and $H^1(\pi(f)^*TZ)$.

Suppose that (Σ, f) has more than one subfactor. Both $\text{Coker } L_{\Sigma, f}^{Y, D}$ and $\text{Coker } L_{\pi(\Sigma, f)}^Z$ are obtained by requiring the residues at the new marked points to be opposite to each other. Then our proof also extends to this case. Then we finish the proof of Lemma 3.4. \square

Next, we continue the proof of Proposition 3.3. Since we have identified the obstruction spaces, we first choose a stabilization term η_i on $\overline{\mathcal{B}}_{\pi_*(A)}^Z(0, k + l + q, J)$ to dominate the local obstruction bundle generated by $\text{Coker } L_{\pi(\Sigma, f)}^Z$. Then, we pull back η_i over $\overline{\mathcal{B}}_{\Gamma_0, T_k}(Y, D, J)$. By Lemma 3.4, it dominates $\text{Coker } L^{Y, D}$. This implies that π induces a smooth map on virtual neighborhood and a commutative diagram on obstruction bundles

$$(14) \quad \begin{array}{ccc} E_{Y, D} & \longrightarrow & E_Z \\ \downarrow & & \downarrow \\ \pi_{\mathcal{S}_e} : U_{\mathcal{S}_e}^{Y, D} & \longrightarrow & U_{\mathcal{S}_e}^Z. \end{array}$$

Furthermore, the proper sections $S_{Y, D}, S_Z$ commutes with the above diagram. $\pi_{\mathcal{S}_e}$ commutes with the evaluation map for those β_i classes. Choose a Thom form Θ of E_Z . Its pullback is the Thom form on $E_{Y, D}$ (still denoted by Θ).

It is clear that

$$\dim D_i = \dim U_{\mathcal{S}_e}^{Y, D} - 2.$$

By our construction, the D_i intersect each other transversely. Note that

$$\begin{aligned} \dim U_{\mathcal{S}_e}^{Y, D} &= \text{rank } E_Z + 2(c_1^Y(A) + n - 3 + l + q + k - \sum t_i). \\ \dim U_{\mathcal{S}_e}^Z &= \text{rank } E_Z + 2(c_1^Z(\pi_*(A)) + n - 1 - 3 + l + q + k). \end{aligned}$$

However,

$$c_1^Y(A) = c_1^Z(\pi_*(A)) + c_1(N)(\pi_*(A)) + 2 \sum t_i = c_1^Z(\pi_*(A)) + Z^*(A) + \sum t_i.$$

Hence,

$$\dim U_{\mathcal{S}_e}^{Y, D} - \dim U_{\mathcal{S}_e}^Z = 2(Z^*(A) + 1).$$

By definition, we have

$$\begin{aligned} & \deg \Theta + \sum_i \deg(\beta_i) + \deg \varpi + \sum_j \deg(\delta_j) \\ (15) \quad &= \dim U_{\mathcal{S}_e}^{Y, D} - 2d > \dim U_{\mathcal{S}_e}^Z, \end{aligned}$$

where $\deg \varpi = \sum \deg \alpha_j$. Then, from (15) and $Z^*(A) \geq d$, we have

$$\begin{aligned} & (S_{Y,D})^* \Theta \prod \psi_i^{d_i-1} ev_i^* \beta_i \wedge ev^* \varpi \wedge \prod ev_j^* \delta_j \\ &= ((S_{Y,D})^* \Theta \prod \psi_i^{d_i-1} (\pi_{S_e})^* (ev_i^* \beta_i \wedge \prod ev^* \varpi \wedge \prod ev_j^* \delta_j)) = 0. \end{aligned}$$

In the last equality, we use $ev_i^* \beta_i \wedge \prod ev^* \varpi \wedge \prod ev_j^* \delta_j = 0$ on $U_{S_e}^Z$. Hence, the relative invariant is zero. This completes the proof of Proposition 3.3. \square

Remark 3.5. *McDuff also proved the same result in the case without insertion classes β_i by a totally different method; see Lemma 1.7 in [M1].*

3.3. A nonvanishing theorem. When we apply the degeneration formula, we often need to compute some special terms where the degeneration graph lies completely on the side of the projective bundle. In the previous subsection, we proved a vanishing theorem for some relative invariants of (Y, D) . In this subsection, we will consider the case where the relative invariants of (Y, D) with empty relative insertion on D are no longer zero and the invariant of Z will contribute in a nontrivial way.

Suppose that $A \in H_2(Z, \mathbb{Z})$. Denote by $i : Z \rightarrow Y$ the embedding of Z into Y as the zero section. Consider the relative invariant of (Y, D)

$$\langle \tau_{i_1}(\beta_1[Z]), \tau_{i_2}(\beta_2[Z]), \dots, \tau_{i_k}(\beta_k[Z]), \varpi \mid \emptyset \rangle_{0,A}^{Y,D},$$

where ϖ consists of insertions of the form $\pi^* \alpha_1, \dots, \pi^* \alpha_l$. The dimension condition is

$$2 \sum (i_t + 1) + \sum \deg \beta_t + \deg \varpi = 2(C_1^Y(A) + n - 3 + k + l).$$

The dimension condition of the divisor invariant $\langle \tau_{i_1}(\beta_1), \dots, \tau_{i_k}(\beta_k), i^* \varpi \rangle_{0,A}^Z$ is

$$2(C_1^Z(A) + n - 1 - 3 + k + l) = 2 \sum i_t + \sum \deg \beta_t + \deg \varpi.$$

Since $C_1^Y(A) = C_1^Z(A) + Z \cdot A$, so both invariants are well-defined only when $k = Z \cdot A + 1$.

Theorem 3.6. *Let $A \in H_2(Z, \mathbb{Z})$. Suppose that $k = Z \cdot A + 1$ and $C_1(L)(C) \geq 0$ for any holomorphic curve C into Z . Then*

$$\begin{aligned} & \langle \varpi, \beta_1 \cdot [Z], \beta_2 \cdot [Z], \dots, \beta_k \cdot [Z] \mid \emptyset \rangle_{0,A}^{Y,D} \\ &= \langle \iota^* \varpi, \beta_1, \dots, \beta_k \rangle_{0,A}^Z, \end{aligned}$$

where ϖ consists of insertions of the form $\pi^* \alpha_1, \dots, \pi^* \alpha_l$ and $\iota : Z \rightarrow Y$ is the embedding of Z into Y as the zero section.

Proof. Choose a Hermitian metric and a unitary connection on L such that they induce a splitting

$$0 \rightarrow V \rightarrow TY \rightarrow \pi^* TZ \rightarrow 0,$$

where V is the vertical tangent bundle. We choose a metric of TY as the direct sum of a metric on V and $\pi^* TZ$, where the second one is induced

from a metric on Z . The Levi-Civita connection is a direct sum. Therefore, we may choose almost complex structures J_Z on TZ and J_V on V such that we may choose the direct sum $J_Z \oplus J_V$ as an almost complex structure J_Y on TY . It is easy to see that $\bar{\partial}$ commutes with π .

From Lemma 3.4, we know that the projection $\pi : Y \rightarrow Z$ induces a smooth map $\pi_{\mathcal{S}_e} : U_{\mathcal{S}_e}^{Y,D} \rightarrow U_{\mathcal{S}_e}^Z$ on virtual neighborhoods and the obstruction bundle $E_{Y,D}$ over $U_{\mathcal{S}_e}^{Y,D}$ is the pullback of the obstruction bundle E_Z over $U_{\mathcal{S}_e}^Z$. Therefore, by the definition of Gromov-Witten invariants, we have

$$(16) \quad \begin{aligned} & \langle \varpi \beta_1 \cdot [Z], \beta_2 \cdot [Z], \dots, \beta_k \cdot [Z] \mid \emptyset \rangle_{0,A}^{Y,D} \\ &= \deg(\pi_{\mathcal{S}_e}) \langle i^* \varpi, \beta_1, \dots, \beta_k \rangle_{0,A}^Z. \end{aligned}$$

We claim that $\deg(\pi_{\mathcal{S}_e}) = 1$.

In fact, by the construction of virtual neighborhoods, we know that for every generic element $(\mathbb{P}^1, x_1, \dots, x_{k+l}, \tilde{f}) \in U_{\mathcal{S}_e}^Z$, there is a section ν of the obstruction bundle E_Z such that $\bar{\partial}_{J_Z} \tilde{f} = \nu$.

Suppose that a generic element $(\mathbb{P}^1, x_1, \dots, x_{k+l}, f) \in U_{\mathcal{S}_e}^{Y,D}$ is a preimage of $(\mathbb{P}^1, x_1, \dots, x_{k+l}, \tilde{f})$ under $\pi_{\mathcal{S}_e}$. That is, f is a lifting of \tilde{f} to Y vanishing at the marked points x_1, \dots, x_k . Therefore, from the fact that $\bar{\partial}$ commutes with π , we have that $(\mathbb{P}^1, x_1, \dots, x_{k+l}, f)$ satisfies

$$(17) \quad \bar{\partial}_{J_Y} f = \pi_{\mathcal{S}_e}^* \nu.$$

If we choose a local coordinate (z, s) on Y , where s is the Euclidean coordinate on the fiber \mathbb{P}^1 , then locally we may write $f = (\tilde{f}, f^V)$. Therefore (17) locally can be written as

$$(18) \quad \begin{cases} \bar{\partial}_{J_Z} \tilde{f} &= \nu, \\ \bar{\partial}_{J_V} f^V &= 0 \end{cases}$$

Since $\bar{\partial}^2 = 0$ always holds on \mathbb{P}^1 , it follows from a well-known fact of complex geometry that f^*L is a holomorphic line bundle over \mathbb{P}^1 . Moreover, (18) shows that f^V gives rise to a holomorphic section of the bundle f^*L , up to \mathbb{C}^* , which vanishes at the marked points x_1, \dots, x_k . Since $\deg(f^*L) = Z \cdot A$, therefore, from our assumption that $k = Z \cdot A + 1$, we know that $f^*L \otimes (-x_1 - \dots - x_k)$ has no nonzero holomorphic sections. Therefore, $f^V \equiv 0$. This says that the only preimage of a generic element $(\mathbb{P}^1, x_1, \dots, x_{k+l}, \tilde{f})$ in $U_{\mathcal{S}_e}^Z$ is itself. This implies $\deg(\pi_{\mathcal{S}_e}) = 1$. This proves the theorem. \square

4. A COMPARISON THEOREM

Let X be a compact symplectic manifold and $Z \subset X$ be a smooth symplectic submanifold of codimension 2. $\iota : Z \rightarrow X$ is the inclusion map. The cohomological push-forward

$$\iota^! : H^*(Z, \mathbb{R}) \rightarrow H^*(X, \mathbb{R})$$

is determined by the pullback ι^* and Poincaré duality.

Definition 4.1. *A symplectic divisor Z is said to be positive if for some tamed almost complex structure J , $C_1(N_Z)(A) > 0$ for any A represented by a non-trivial J -sphere in Z .*

This is a generalization of ample divisor from algebraic geometry. Define (19)

$$V := \min\{C_1(N_{Z|X})(A) > 0 \mid A \in H_2(Z, \mathbb{Z}) \text{ is a stably effective class}\}.$$

In [MP], the authors point out that the relative Gromov-Witten theory of (X, Z) does not provide new invariants: the relative Gromov-Witten theory of (X, Z) is completely determined by the absolute Gromov-Witten theory of X and Z in principle. In this section, under some positivity assumptions on the normal bundle of the divisor, we will give an explicit relation between the absolute and relative Gromov-Witten invariants, which we call as a comparison theorem. The main tool of this section is the degeneration formula of Gromov-Witten invariants. The central theorem of this section is

Theorem 4.2. *Suppose that Z is a positive divisor and $V \geq l$. Then for $A \in H_2(X, \mathbb{Z})$, $\alpha_i \in H^*(X, \mathbb{R})$, $1 \leq i \leq \mu$, and $\beta_j \in H^*(Z, \mathbb{R})$, $1 \leq j \leq l$, we have*

$$\begin{aligned} & \langle \alpha_1, \dots, \alpha_\mu, \iota^!(\beta_1), \dots, \iota^!(\beta_l) \rangle_A^X \\ &= \sum_{\mathcal{T}} \langle \alpha_1, \dots, \alpha_\mu \mid \mathcal{T} \rangle_A^{X, Z}, \end{aligned}$$

where the summation runs over all possible weighted partitions $\mathcal{T} = \{(1, \gamma_1), \dots, (1, \gamma_q), (1, [Z]), \dots, (1, [Z])\}$ where γ_i 's are the products of some β_j classes.

Proof. We perform the symplectic cutting along the boundary of a tubular neighborhood of Z . Then we have $\overline{X}^- = X$, $\overline{X}^+ = \mathbb{P}(N_{Z|X} \oplus \mathbb{C})$. Since $\beta_i \in H^*(Z, \mathbb{R})$, we choose the support of $\iota^!(\beta_i)$ near Z . Then, $\iota^!(\beta_i)^- = 0$, $\iota^!(\beta_i)^+ = \iota^!(\beta_i)$. Here ι in the second term is understood as the inclusion map of Z via the zero section into $Y = \mathbb{P}(N_{Z|X} \oplus \mathbb{C})$. Up to a rational multiple, each α_i is Poincaré dual to an immersed submanifold W_i . We can perturb W_i to be transverse to Z . In a neighborhood of Z , W_i is $\pi^{-1}(W_i \cap Z)$, where $\pi : N_{Z|X} \rightarrow Z$ is the projection. Clearly, π induces the projection $\mathbb{P}(N_{Z|X} \oplus \mathbb{C}) \rightarrow Z$, still denoted by π . The symplectic cutting naturally decomposes W_i into $W_i^- = W_i$, $W_i^+ = h^*(\alpha_i|_Z) = h^*\tilde{\alpha}_i$. In other words, we can choose $\alpha_i^- = \alpha_i$, $\alpha_i^+ = h^*(\alpha_i|_Z)$.

Now we apply the degeneration formula for invariants

$$\langle \alpha_1, \dots, \alpha_\mu, \iota^!(\beta_1), \dots, \iota^!(\beta_l) \rangle_A^X$$

and express it as a summation of products of relative invariants of (X, Z) and (Y, D) . Moreover, from the degeneration formula, each summand Ψ_C may consist of a product of relative Gromov-Witten invariants with disconnected domain curves of both (X, Z) and (Y, D) .

On the side of Y , there may be several disjoint components. Let A' be the total homology class. Then, from our assumption, we have $Z \cdot A' = Z \cdot A \geq d$. Suppose that we have a nonzero summand $\Psi_C \neq 0$. We claim that each factor from the relative Gromov-Witten invariants of (Y, D) must be in the form $\langle \iota^!(\beta_{i_1}), \dots, \iota^!(\beta_{i_t}) | (1, \gamma) \rangle_F^{Y, D}$ or $\langle |(1, [pt])| \rangle_F^{Y, D}$, where F is the homology class of a fiber of Y and γ is a basis element of $H^*(D, \mathbb{R})$ such that $\int_Z \beta_{i_1} \wedge \dots \wedge \beta_{i_t} \wedge \gamma \neq 0$. Note that for these components, $Z^*(sF) = s$. From our assumption $V \geq d$ and Proposition 3.3, the nonzero factor of the relative Gromov-Witten invariants of (Y, D) must be the fiber class relative invariants. From Proposition 3.2, we know that the nonzero factor must be of the form $\langle \iota^!(\beta_{i_1}), \dots, \iota^!(\beta_{i_t}) | (1, \gamma) \rangle_F^{Y, D}$ or $\langle |(1, [pt])| \rangle_F^{Y, D}$. Moreover, if some $\beta_i = [pt]$, then the marked point must be in a two-point component and the nonzero relative invariant must be $\langle \iota^!([pt]) | (1, [Z]) \rangle_F^{Y, D} = 1$.

Since there are no vanishing two-cycles in this case, we may write down the summation as follows.

$$\begin{aligned} & \langle \alpha_1, \dots, \alpha_\mu, \iota^!(\beta_1), \dots, \iota^!(\beta_l) \rangle_A^X \\ &= \sum_{\mathcal{T}} \langle \alpha_1, \dots, \alpha_\mu \mid \mathcal{T} \rangle_A^{X, Z}, \end{aligned}$$

where the summation runs over all possible weighted partitions $\mathcal{T} = \{(1, \gamma_1), \dots, (1, \gamma_q), (1, [Z]), \dots, (1, [Z])\}$ where γ_i' are the product of some β_j classes. We complete the proof of our comparison theorem. \square

Corollary 4.3. *Under the assumption of Theorem 4.2. If the product of any two β_j classes vanishes, then we have*

$$\langle \alpha_1, \dots, \alpha_\mu, \iota^!(\beta_1), \dots, \iota^!(\beta_l) \rangle_A^X = \langle \alpha_1, \dots, \alpha_l \mid \mathcal{T} \rangle_A^{X, Z},$$

where $\mathcal{T} = \{(1, \beta_1), \dots, (1, \beta_l), (1, [Z]), \dots, (1, [Z])\}$ is a weighted partition of $Z \cdot A$.

5. RATIONALLY CONNECTED SYMPLECTIC DIVISORS

5.1. Rationally connectedness in algebraic geometry. The basic reference for this subsection is [A]. We refer to [C, D, K, KMM1, KMM2, V] for more details.

Let us recall the notion of rational connectedness in algebraic geometry.

Definition 5.1. *Let X be a smooth complex projective variety of positive dimension. We say that X is rationally connected if one of the following equivalent conditions holds.*

- (1) *Any two points of X can be connected by a rational curve (called as rationally connected).*
- (2) *Two general points of X can be connected by a chain of rational curves (called as rationally chain-connected).*
- (3) *Any finite set of points in X can be connected by a rational curve.*

- (4) Two general points of X can be connected by a very free rational curve. Here we say that a rational curve $C \subset X$ is a very free curve if there is a surjective morphism $f : \mathbb{P}^1 \rightarrow C$ such that

$$f^*X \cong \bigoplus_{i=1}^{\dim X} \mathcal{O}_{\mathbb{P}^1}(a_i), \quad \text{with all } a_i \geq 1.$$

Next let us look at some properties of rationally connected varieties.

Proposition 5.2. *The following properties of rationally connected manifolds hold:*

- (1) *Rationally connectedness is a birational invariant.*
- (2) *Rationally connectedness is invariant under smooth deformation.*
- (3) *If X is rationally connected, then $H^0(X, (\Omega_X^1)^{\otimes m}) = 0$ for every $m \geq 1$.*
- (4) *Fano varieties (i.e., smooth complex projective varieties X for which $-K_X$ is ample) are rationally connected. In particular, smooth hypersurfaces of degree d in \mathbb{P}^n are rationally connected for $d \leq n$.*
- (5) *Rationally connected varieties are well behaved under fibration, i.e.: Let X be a smooth complex projective variety. Assume that there exists a surjective morphism $f : X \rightarrow Y$ with Y and the general fiber of f rationally connected. Then X is rationally connected.*

An important theorem connecting birational geometry to Gromov-Witten theory is the result of Kollár and Ruan [K], [R1]: a uniruled projective manifold has a nonzero genus zero GW-invariant with a point insertion. A longstanding problem in Gromov-Witten theory is that a similar result with two point insertions should also hold for rationally connected projective manifolds.

5.2. Rationally connected symplectic divisors. In this subsection, we want to apply our comparison theorem to study the k -point rationally connectedness properties. We will show that a symplectic manifold is k -point strongly rationally connected if it contains a k -point strongly rationally connected symplectic divisor with sufficiently positive normal bundle. Our main tool is the degeneration formula of Gromov-Witten invariants and our comparison Theorem 4.2.

Before we state our main theorem, we want first to define the notion of k -point rational connectedness.

Definition 5.3. *Let $A \in H_2(X, \mathbb{Z})$ be a nonzero class. A is said to be a k -point rationally connected class if there is a nonzero Gromov-Witten invariant*

$$(20) \quad \langle \tau_{d_1}[pt], \dots, \tau_{d_k}[pt], \tau_{d_{k+1}}\alpha_{k+1}, \dots, \tau_{d_l}\alpha_l \rangle_A^X,$$

where $\alpha_i \in H^*(X, \mathbb{R})$ and d_1, \dots, d_l are non-negative integers. We call a class $A \in H_2(X, \mathbb{Z})$ a k -point strongly rationally connected class if $d_i = 0$, $1 \leq i \leq l$, in (20).

Definition 5.4. X is said to be (symplectic) k -point (strongly) rationally connected if there is a k -point (strongly) rationally connected class. We simply call a 2-point (strongly) rationally connected symplectic manifold as (strongly) rationally connected symplectic manifold.

Remark 5.5. From the definition of uniruledness of [HLR], a 1-point rationally connected symplectic manifold is equivalent to a uniruled symplectic manifold. From the definitions, we know that a k -point (strongly) rational connected symplectic manifold must be uniruled.

Remark 5.6. It is possible that k -point rational connectedness is equivalent to k -point strongly rational connectedness. We do not know how to prove this.

Example 5.7. It is well-known that for any positive integer k , the projective space \mathbb{P}^n is k -point strongly rationally connected.

Example 5.8. Let $G(k, n)$ be the Grassmannian manifold of k -planes in \mathbb{C}^n . It is well-known that the classical cohomology of $Gr(k, n)$ has a basis of Schubert classes σ_λ , as λ varies over partitions whose Young diagram fits in a k by $n - k$ rectangle. The (complex) codimension of σ_λ is $|\lambda| = \sum \lambda_i$, the number of boxes in the Young diagram. The quantum cohomology of the Grassmannian is a free module over the polynomial ring $\mathbb{Z}[q]$, with a basis of Schubert classes; the variable q has (complex) degree n . The quantum product $\sigma_\lambda \star \sigma_\mu$ is a finite sum of terms $q^d \sigma_\nu$, the sum over $d \geq 0$ and $|\nu| = |\lambda| + |\mu| - dn$, each occurring with a nonnegative coefficient (a Gromov-Witten invariant). Denote by $\rho = \sigma_{((n-k)^k)}$ the class of a point. In [BCF], the authors proved that $\sigma_\rho \star \sigma_\rho = q^k \sigma_{((n-2k)^k)}$ if $k \leq n - k$, and $\sigma_\rho \star \sigma_\rho = q^{n-k} \sigma_{((n-k)^{2k-n})}$ if $n - k \leq k$. This means that the Grassmannian $Gr(k, n)$ is symplectic rationally connected.

Example 5.9. For any integer $d \geq 0$, Consider the Grassmannian $G(d, 2d)$. Buch-Kresch-Tamvakis [BKT] proved that for three points $U, V, W \in G(d, 2d)$ pairwise in general position, there is a unique morphism $f : \mathbb{P}^1 \rightarrow G(d, 2d)$ of degree d such that $f(0) = U$, $f(1) = V$ and $f(\infty) = W$. This implies that the Gromov-Witten invariant $\langle [pt], [pt], [pt] \rangle_d^{G(d, 2d)} = 1$. Therefore, $G(d, 2d)$ are 3-point strongly rationally connected.

Example 5.10. Let $\mathbb{H} = \text{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$ the Hilbert scheme of points on $\mathbb{P}^1 \times \mathbb{P}^1$. In [P], the author gave a \mathbb{Q} -basis for $A^*(\mathbb{H})$ as follows: $T_0 = [\mathbb{H}]$, $T_1, T_2, T_3, T_4, T_5 = T_1 T_2, T_6 = T_1^2, T_7 = T_2^2, T_8 = T_1 T_3, T_9 = T_2 T_3, T_{10} = C_2 + F, T_{11} = C_1 + F, T_{12} = C_1 + C_2 + F$ and T_{13} the class of a point. The author also computed the quantum product $T_4 \star T_4 = T_{13} + 2q_1 q_2 q_3^2 T_0$ and $T_4 \star T_4 \star T_4 = 2q_1 q_2 q_3^2 T_4$. So we have $T_{13} \star T_{13} = -2q_1 q_2 q_3^2 T_{13} \neq 0$. This implies that \mathbb{H} is symplectic rationally connected. From the computation of [G], it is easy to know that $\text{Hilb}^2(\mathbb{P}^2)$ also is symplectic rationally connected.

Since Gromov-Witten invariants are invariant under smooth symplectic deformations, k -point rational connectedness is invariant under smooth symplectic deformations. It is not yet known whether a projective rationally connected manifold is symplectic rationally connected. Moreover, so far, we could not show that this notion is invariant under symplectic birational cobordisms defined in [HLR]. We will leave this for future research. However, we would like to mention some partial results along this direction. From the blowup formula of Gromov-Witten invariants in [H1, H2, H3, HZ, La], we have

Proposition 5.11. *Suppose that X is a k -point strongly rationally connected symplectic manifold. Let \tilde{X} be the blowup of X along a finite number of points or some special submanifolds with convex normal bundles (see, [H1, La]). Then \tilde{X} is k -point strongly rationally connected.*

In 1991, McDuff [M3] first observed that a semi-positive symplectic 4-manifold, which contains a submanifold P symplectomorphic to \mathbb{P}^1 whose normal Chern number is non-negative, must be uniruled. In [LtiR], the authors generalize McDuff's result to more general situations. More importantly, they gave a rather general *from divisor to ambient space* inductive construction of uniruled symplectic manifolds. In this subsection, we will generalize their inductive construction to the case of rationally connected symplectic manifolds.

Suppose that X is a compact symplectic manifold and $Z \subset X$ is a symplectic submanifold of codimension 2. Denote by $N_{Z|X}$ the normal bundle of Z in X . Denote by $\iota : Z \subset X$ the inclusion of Z into X . Let V be the minimal normal Chern number defined in (19). We call a class $A \in H_2(Z, \mathbb{R})$ a **minimal class** if $Z \cdot A = V$.

Theorem 5.12. *Suppose that X is a compact symplectic manifold and $Z \subset X$ is a symplectic submanifold of codimension 2. If Z is k -point strongly rationally connected and $A \in H_2(Z, \mathbb{Z})$ is a minimal class such that*

$$(21) \quad \langle \iota^* \alpha_1, \dots, \iota^* \alpha_l, [pt], \dots, [pt], \beta_{k+1}, \dots, \beta_r \rangle_A^Z \neq 0$$

for some $r \leq V + 1$, $\beta_i \in H^(Z, \mathbb{R})$ and $\alpha_j \in H^*(X, \mathbb{R})$, then X is k -point strongly rationally connected. In particular, if $\iota : Z \rightarrow X$ is homologically injective, then X is k -point strongly symplectic rational connected if $k \leq V + 1$.*

Proof. Since we can always increase the number of Z -insertions by adding divisor insertions in (21), therefore, without loss of generality, we may assume that $r = Z \cdot A + 1$. Consider the following Gromov-Witten invariant of X :

$$(22) \quad \langle \alpha_1, \dots, \alpha_l, [pt], \dots, [pt], \beta_{k+1} \cdot [Z], \dots, \beta_r \cdot [Z] \rangle_A^X.$$

If the invariant (22) is nonzero, then we are done. So in the following we assume that the invariant (22) equals zero.

To find a nonzero Gromov-Witten invariant of X with at least k point insertions, we first apply the degeneration formula to the invariant (22) to obtain a nonzero relative Gromov-Witten invariant of (X, Z) with at least k point insertions, then use our comparison theorem to obtain a nonzero Gromov-Witten invariant of X .

We perform the symplectic cutting along the boundary of a tubular neighborhood of Z . Then we have $\overline{X}^- = X$, $\overline{X}^+ = Y = \mathbb{P}(N_{Z|X} \oplus \mathbb{C})$. Since $\beta_i \in H^*(Z, \mathbb{R})$, we choose the support of $[pt] \cdot Z$, $\beta_i \cdot Z$ near Z . Then, $([pt])^- = 0$, $(\beta_i \cdot Z)^- = 0$, $([pt])^+ = [pt]$, $(\beta_i \cdot Z)^+ = \iota^!(\beta_i)$. Here ι in the second term is understood as the inclusion map of Z via the zero section into $Y = \mathbb{P}(N_{Z|X} \oplus \mathbb{C})$. Up to a rational multiple, each α_i is Poincaré dual to an immersed submanifold W_i . We can perturb W_i to be transverse to Z . In a neighborhood of Z , W_i is $\pi^{-1}(W_i \cap Z)$, where $\pi : N_{Z|X} \rightarrow Z$ is the projection. Clearly, π induces the projection $\mathbb{P}(N_{Z|X} \oplus \mathbb{C}) \rightarrow Z$, still denoted by π . The symplectic cutting naturally decomposes W_i into $W_i^- = W_i$, $W_i^+ = h^*(\alpha_i|_Z) = h^*\tilde{\alpha}_i$. In other words, we can choose $\alpha_i^- = \alpha_i$, $\alpha_i^+ = h^*(\alpha_i|_Z)$.

Now we apply the degeneration formula for the invariant

$$\langle \alpha_1, \dots, \alpha_l, [pt], \dots, [pt], \beta_{k+1} \cdot [Z], \dots, \beta_r \cdot [Z] \rangle_A^X.$$

and express it as a summation of products of relative invariants of (X, Z) and (Y, D) . Moreover, from the degeneration formula, each summand Ψ_C may consist of a product of relative Gromov-Witten invariants with disconnected domain curves of both (X, Z) and (Y, D) .

On the side of Y , there may be several disjoint components. We claim that every component on the side of Y is a multiple of the fiber class. Suppose that there is a component on the side of Y which has a homology class $A' + \mu F$ where $A' \in H_2(Z, \mathbb{Z})$ and F is the fiber class. Then we have $Z \cdot (A' + \mu F) = Z \cdot A' + \mu$. Since every component must intersect the infinity section of Y , $\mu > 0$. Therefore, we have $Z \cdot (A' + \mu F) \geq V + 1$. From Proposition 3.3, we know that the contribution of this component to the corresponding relative Gromov-Witten invariants of (Y, D) must be zero. So the corresponding summand in the degeneration formula must be zero. From Proposition 3.2 and the same argument as in the proof of Theorem 4.2, we have

$$\begin{aligned} 0 &= \langle \alpha_1, \dots, \alpha_l, [pt], \dots, [pt], \beta_{k+1} \cdot [Z], \dots, \beta_r \cdot [Z] \rangle_A^X \\ &= \sum_{\mu} C_{\mu} \langle \alpha_1, \dots, \alpha_l \mid \mu \rangle_A^{X, Z} \\ &\quad + \langle i^* \alpha_1, \dots, i^* \alpha_l, [pt], \dots, [pt], \beta_{k+1} \cdot [Z], \dots, \beta_r \cdot [Z] \mid \emptyset \rangle_A^{Y, D} \\ &= \sum_{\mu} C_{\mu} \langle \alpha_1, \dots, \alpha_l \mid \mu \rangle_A^{X, Z} \\ &\quad + \langle i^* \alpha_1, \dots, i^* \alpha_l, [pt], \dots, [pt], \beta_{k+1}, \dots, \beta_r \rangle_A^Z, \end{aligned}$$

where we used Theorem 3.6 in the last equality and the summation runs over the possible partitions $\mu = \{(1, [pt]), \dots, (1, [pt]), (1, \gamma_1), \dots, (1, \gamma_q), (1, [Z]), \dots, (1, [Z])\}$ where γ_i are the product of some β_i classes.. From our assumption (21), we have

$$(23) \quad \sum_{\mu} C_{\mu} \langle \alpha_1, \dots, \alpha_l \mid \mu \rangle_A^{X,Z} \neq 0.$$

Denote by $\langle \alpha_1, \dots, \alpha_l \mid \mu_0 \rangle_A^{X,Z}$ the minimal nonzero relative invariant in the summand (23) in the sense of Definition 2.11. Write $\mu_0 = \{(1, [pt]), \dots, (1, [pt]), (1, \gamma_1), \dots, (1, \gamma_q), (1, [Z]), \dots, (1, [Z])\}$. Then it is easy to know that the product of any two γ_i and γ_j vanishes.

Now we consider the following absolute Gromov-Witten invariant

$$\langle \alpha_1, \dots, \alpha_l, [pt], \dots, [pt], \gamma_1 \cdot [Z], \dots, \gamma_l \cdot [Z], [Z], \dots, [Z] \rangle_A^X.$$

Applying the degeneration formula and always distribute the point insertions to the side of (Y, D) . Therefore, from Theorem 4.2, we have

$$\begin{aligned} & \langle \alpha_1, \dots, \alpha_l, [pt], \dots, [pt], \gamma_1 \cdot [Z], \dots, \gamma_q \cdot [Z] \rangle_A^X \\ &= \langle \alpha_1, \dots, \alpha_l \mid \mu_0 \rangle_A^{X,Z} \neq 0. \end{aligned}$$

This implies that X is k -point strongly rationally connected. This proves our theorem. \square

It is well-known that \mathbb{P}^{n-1} is strongly rationally connected. Therefore, from Theorem 5.12, we have

Corollary 5.13. *Let (X, ω) be a compact $2n$ -dimensional symplectic manifold which contains a submanifold P symplectomorphic to \mathbb{P}^{n-1} whose normal Chern number $m \geq 2$. Then X is strongly rationally connected.*

REFERENCES

- [A] C. Araujo, Rationally connected varieties, arXiv:math.AG/0503305
- [BCF] A. Bertram, I. Ciocan-Fontanine, W. Fulton, Quantum multiplication of Schur Polynomials, J. Alg., 219(1999), 728-746.
- [BKT] A. S. Buch, A. Kresch, H. Tamvakis, Gromov-Witten invariants on Grassmannians, J. Amer. Math. Soc., 16(4)(2003), 901-915.
- [C] F. Campana, Connexit   tationnelle des vari  t  s de Fano, Ann. Sci.   cole Norm. Sup. 25(4)(1992), no.5, 539-545.
- [D] O. Debarre, Higher-dimensional algebraic geometry, Universitext, Springer-Verlag, New York, 2001.
- [G] T. Graber, Enumerative geometry of hyperelliptic plane curves, J. Alg. Geom., 10(2001), 725-755.
- [H1] J. Hu, Gromov-Witten invariants of blow-ups along points and curves, Math. Z. 233(2000), 709-739.
- [H2] J. Hu, Gromov-Witten invariants of blow-ups along surfaces, Compositio Math., 125(2001), 345-352.
- [H3] J. Hu, Gromov-Witten invariants of blowups along semi-positive submanifold, Adv. in Math. Research, vol. 3(2003), 103-109.

- [HLR] J. Hu, T.-J. Li, Y. Ruan, Birational cobordism invariance of uniruled symplectic manifolds, math.SG/0611592, to appear in Invent. Math.
- [HZ] J. Hu, H. Zhang, Elliptic Gromov-Witten invariants of blowups along surfaces, Int. J. Math. Math. Sci, 2005:1(2005), 81-90.
- [IP] E. Ionel, T. Parker, Relative Gromov-Witten invariants, Ann. of Math., 157(2)(2003), 45-96.
- [K] J. Kollár, Rationally curves on algebraic varieties, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 32, Springer-Verlag, Berlin, 1996.
- [KMM1] J. Kollár, Y. Miyaoka, S. Mori, Rational connectedness and boundedness of Fano manifolds, J. Differential Geom. 36(3)(1992), 765-779.
- [KMM2] J. Kollár, Y. Miyaoka, S. Mori, Rationally connected varieties, J. Algebraic Geom. 1(3)(1992), 429-448.
- [La] H. Lai, Gromov-Witten invariants of blowups along submanifolds with convex normal bundles, arXiv:math.AG/0710.3968.
- [L] E. Lerman, Symplectic cuts, Math. Research Lett. 2(1995), 247-258.
- [Li1] J. Li, Stable morphisms to singular schemes and relative stable morphisms, J. Diff. Geom. 57(2001), 509-578.
- [Li2] J. Li, Relative Gromov-Witten invariants and a degeneration formula of Gromov-Witten invariants, J. Diff. Geom. 60(2002), 199-293
- [LR] A. Li, Y. Ruan, Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds, Invent. Math. 145(2001), 151-218.
- [LjtR] T.-J. Li, Y. Ruan, Uniruled symplectic divisors, arXiv:math.SG/0711.4254
- [MP] D. Maulik, R. Pandharipande, A topological view of Gromov-Witten theory, Topology, 45(5)(2006), 887-918.
- [M1] D. McDuff, Comparing absolute and relative Gromov-Witten invariants, preprint.
- [M2] D. McDuff, Hamiltonian S^1 -manifolds are uniruled, preprint, 2007
- [M3] D. McDuff, Symplectic manifolds with contact type boundaries, Invent. Math. 103(1991), 651-671.
- [OP] A. Okounkov, R. Pandharipande, Gromov-Witten theory, Hurwitz numbers, and completed cycles, Ann. of Math., 163(2)(2006), 517-560.
- [P] D. Pontoni, Quantum cohomology of $\text{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$ and enumerative applications, Trans. Amer. Math. Soc., 359(2007), 5419-5448.
- [Q] Z. Qin, private communication.
- [R1] Y. Ruan, Virtual neighborhoods and pseudoholomorphic curves, Turkish J. Math., 23(1999), 161-231.
- [R2] Y. Ruan, Surgery, Quantum cohomology and birational geometry, Northern California Symplectic Geometry Seminar, 183-198, Amer. Amth. Soc. Transl. Ser.2, 196. Amer. Math. Soc., Providence, RI, 1999.
- [R3] Y. Ruan, Topological sigma model and Donaldson type invariants in Gromov theory, Duke Math. J. 83(2)(1996), 461-500.
- [S1] N. Siebert, Gromov-Witten invariants for general symplectic manifolds, New trends in algebraic geometry(Warwick, 1996), 375-424, Cambridge Uni. Press, 1999.
- [V] C. Voisin, Rationally connected 3-folds and symplectic geometry, arXiv:0801.1396.

DEPARTMENT OF MATHEMATICS, ZHONGSHAN UNIVERSITY, GUANGZHOU, P. R. CHINA
E-mail address: stsjsxu@mail.sysu.edu.cn

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109-1109, AND, YANGTZE CENTER OF MATHEMATICS, SICHUAN UNIVERSITY, CHENGDU, 610064, P.R. CHINA
E-mail address: ruan@umich.edu